Curriculum Burst 63: An Array of Sums

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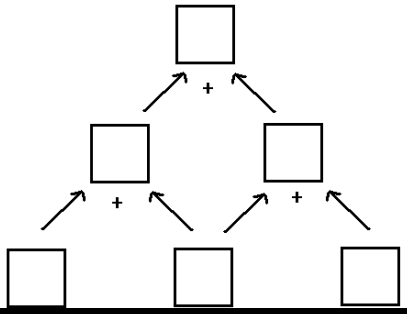
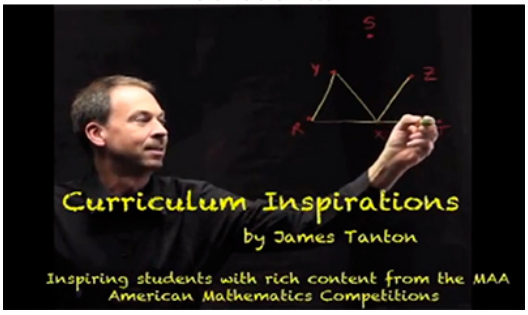
Three different one-digit positive integers are placed in the bottom row of cells. Numbers in adjacent cells are added and the sum is placed in the cell above them. In the second row, continue the same process to obtain a number in the top cell. What is the difference between the largest and smallest numbers possible in the top cell?



QUICK STATS:

Click here for video



MAA AMC GRADE LEVEL

This question is appropriate for the 8th grade level.

MATHEMATICAL TOPICS

Algebra

COMMON CORE STATE STANDARDS

7-EE.4 Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.

PROBLEM SOLVING STRATEGY

ESSAY 2: DO SOMETHING

SOURCE: This is question # 22 from the 2006 MAA AMC 8 Competition.

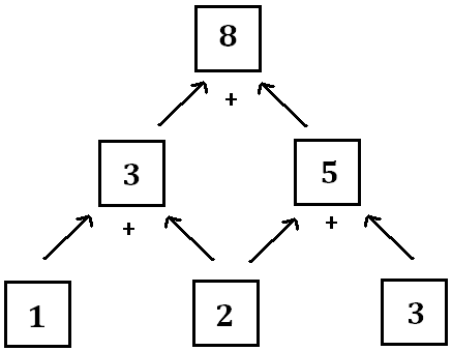
THE PROBLEM-SOLVING PROCESS:

As always, start with …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question seems unusual – and fun! We put some numbers in the bottom boxes and they combine via additions to produce a number at the top.

Just to get a feel for it, putting 1 and 2 and 3 into the bottom gives 8 at the top.



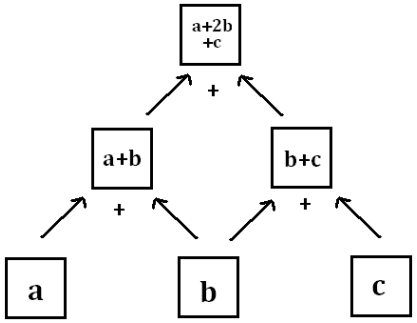
We want to find the biggest and smallest numbers we can produce at the top.

To get the smallest number at the top I am guessing we need to put the smallest numbers in the bottom. Putting in $1, 1, 1$ gives 4 at top – but I need to keep in mind that the question wants three <u>different</u> one-digit numbers to go into the machine.

So, maybe putting in $1, 2, 3$, the three smallest distinct one-digit numbers, gives the smallest output. In the above picture that output is 8. But something is making me hesitate. Is 8 really the smallest possible output? In the above picture the number 2 is "used" twice to get the numbers in the second row. What if 1 was in the middle position instead?

[Try it! Put $2,1,3$ into the machine and show that this gives a top output of 7 !]

Hmm. What's going on in general? Let's be abstract and put in three numbers a, b, c into the machine.



We see that the top output is $a + 2b + c$. Okay, for this to be as small as possible using only one digit numbers, we need b to be as small as it can be, $b = 1$, and then a and c as small as they can be: $a = 2$, $c = 3$. This means that 7 is indeed the smallest possible output.

Also, to make $a + 2b + c$ as large as it can be, use $b = 9$, $a = 8$ and $c = 7$. The largest output is 33.

To answer the question: The difference of the largest and smallest possible outputs is $33 - 7 = 25$.

Extension 1: Again using only distinct one-digit inputs, what is the difference of the largest and smallest possible outputs of an analogous "addition machine" with four, instead of three, boxes at its base? Five boxes? Six? Seven?

Extension 2: Replace all the addition signs with subtraction signs (and allow negative answers in the boxes). Again using distinct one-digit positive numbers as inputs, what is the difference of the largest and smallest possible outputs of a "subtraction machine" which three boxes at its base? (Four boxes? Five? ….)

Curriculum Inspirations is brought to you by the Mathematical Association of America and the MAA American Mathematics Competitions.

MAA acknowledges with gratitude the generous contributions of
the following donors to the Curriculum Inspirations Project:

The TBL and Akamai Foundations for providing continuing support

The Mary P. Dolciani Halloran Foundation for providing seed funding by
supporting the Dolciani Visiting Mathematician Program during fall 2012

MathWorks for its support at the Winner's Circle Level