Curriculum Burst 57: Logs of Divisors

By Dr. James Tanton, MAA Mathematician in Residence

> The sum of the base-10 logarithms of the divisors of 10^n is 792. What is n ?

QUICK STATS:

Click here for video

MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

MATHEMATICAL TOPICS

Logarithms, Summation

COMMON CORE STATE STANDARDS

F-BF.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 1: **ENGAGE IN SUCCESSFUL FLAILING**

SOURCE: This is question # 23 from the 2008 MAA AMC 12B Competition.

THE PROBLEM-SOLVING PROCESS:

As always, the best first step is …

```
STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
```

This question feels scary! I know the words in the question – logarithm, divisor (that is, factor) – and I know what I need to do – list all the factors of 10^n, take their logarithms, add, and set this sum equal to 792.But that task seems overwhelming and potentially very ugly. Hmm.

Let's engage in the art of "successful flailing." Let's follow the start we have and just see how it goes. If it is too awkward it might nonetheless shed light on an alternative route to follow.

Okay. We need to find the divisors of 10^n and compute their logarithms. What are the factors of 10^n ?

Well $10^n = 2^n 5^n$ and any factor must be a product of a power of 2 and a power of 5. The factors are thus the numbers of the form $2^a \cdot 5^b$ with a and b each integers between 0 and n. The list of factors is:

$1,\ 2, 5,\ 2^2, 2\cdot 5, 5^2,\ 2^3, 2^2\cdot 5, 2\cdot 5^2, 5^3,\ \ldots, 2^n 5^n.$

Alright, so let's now take their logarithms. That's not too bad: $\log\left(2^a 5^b\right) = a\log(2) + b\log(5)$.

Next is to add all these logarithms – and that's the part that feels hard. Let's just start writing it out and see if any structure to the sum becomes clear.

Following the list of factors above we get the sum:

$$\begin{aligned}&\log(1)+\log(2)+\log(5)+2\log(2)\\ &\quad+\log(2)+\log(5)+2\log(5)\\ &\quad+3\log(2)+2\log(2)+\log(5)\\ &\qquad+\log(2)+2\log(5)+3\log(5)+\cdots\end{aligned}$$

This looks like a mess! We need to count how many $\log(2)$s are in this sum and how many $\log(5)$s. Hmm. That seems hard.

Maybe the problem is how I chose to list my factors. Let's rewrite the list so that is more focused on the 2s.

We have the factors: $1, 2, 2^2, 2^3, \ldots, 2^n$,

and the factors : $5,\ 2\cdot 5, 2^2\cdot 5, 2^3\cdot 5, \ldots, 2^n\cdot 5$,

and the factors: $5^2, 2\cdot 5^2, 2^2\cdot 5^2, 2^3\cdot 5^2, \ldots, 2^n\cdot 5^2$,

all the way up to : $5^n, 2\cdot 5^n, 2^2\cdot 5^n, 2^3\cdot 5^n, \ldots, 2^n\cdot 5^n$.

Now the sum of logarithms in the first list is:

$$\begin{aligned}&\log(1)+\log(2)+2\log(2)+\cdots+n\log(2)\\ &=0+\log(2)(1+2+\cdots+n)=\frac{1}{2}n(n+1)\log(2)\end{aligned}$$

(See http://www.jamestanton.com/?p=1006 to explain the summation used here.) The sum of logarithms for the second list is:

$$\begin{aligned}&\log(5)+\log(5)+\log(2)\\ &\quad+\log(5)+2\log(2)+\cdots+\log(5)+n\log(2)\\ &\qquad=(n+1)\log(5)+\frac{1}{2}n(n+1)\log(2)\end{aligned}$$

And for the third list it is: $(n+1)2\log(5)+\frac{1}{2}n(n+1)\log(2)$.

And so on up to the final list which has logarithm sum:

$$(n+1)n\log(5)+\frac{1}{2}n(n+1)\log(2).$$

I am not sure where this is going (it still looks messy), but the total sum of logarithms is the sum of all these answers and this is:

$$(n+1)\log(5)(1+2+\cdots+n)+(n+1)\frac{1}{2}n(n+1)\log(2).$$

We can rewrite this as:

$$\begin{aligned}&(n+1)\log(5)\frac{1}{2}n(n+1)+(n+1)\frac{1}{2}n(n+1)\log(2)\\ &=\frac{1}{2}n(n+1)^2\left(\log(5)+\log(2)\right)\\ &=\frac{1}{2}n(n+1)^2\log(10)=\frac{1}{2}n(n+1)^2.\end{aligned}$$

Whoa! All we need do now is set this equal to 792 and see what n is!

$$n(n+1)^2 = 2\cdot 792.$$

Oh heavens! We need to solve a cubic equation? Perhaps there is something special about 792 ? Since $7+2=9$ I see that $792 = 11\times 72$, so

$$2\cdot 792 = 2\times 11\times 72 = 11\times 144 = 11\times 12^2.$$

That's it. We have $n = 11$. Crazy!

Extension: Let $d(N)$ be the count of factors of the number N. Prove that the product of the divisors of N equals $N^{d(N)/2}$.

Curriculum Inspirations is brought to you by the [Mathematical Association of America](#) and the [MAA American Mathematics Competitions](#).