Curriculum Burst 56: Bug Walking

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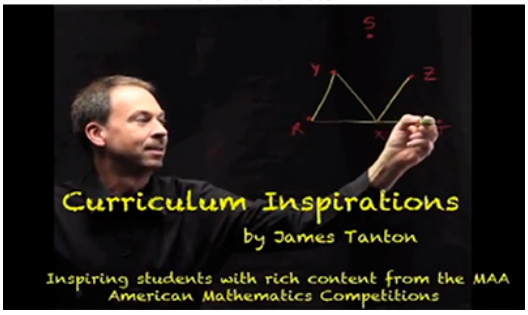
A bug starts at one vertex of a cube and moves along the edges of the cube according to the following rule. At each vertex the bug will choose to travel along one of the three edges emanating from that vertex. Each edge has equal probability of being chosen, and all choices are independent. What is the probability that after seven moves the bug will have visited every vertex exactly once?

QUICK STATS:

Click here for video



MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

MATHEMATICAL TOPICS

Probability

COMMON CORE STATE STANDARDS

S-CP.B Use the rules of probability to compute the probabilities of compound events in a uniform probability model

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 4: **DRAW A PICTURE**

SOURCE: This is question # 20 from the 2006 MAA AMC 12A Competition.

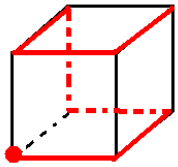
THE PROBLEM-SOLVING PROCESS:

As always, the appropriate first step is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question feels a tad overwhelming as it is three-dimensional (and so "seeing" in my mind's eye what is happening is hard) and there likely to be many many many different paths the bug could walk and keeping track of them seems daunting.

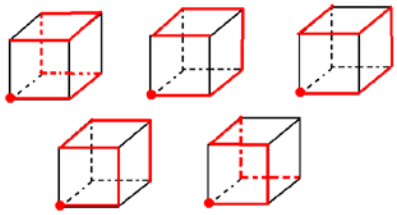
But we need to start somewhere. Let's just draw a picture of a cube and a path the bug could walk.



Actually this was helpful! I can see now that since there are eight vertices to a cube we need seven steps to reach them all. Also, each step must be to a new vertex (and so the bug is not permitted to trace the same edge twice).

One thing: I've started the bug at the bottom left front corner. I don't think the question cares where the bug starts.

Let's draw some more paths that visit each vertex – just to get a feel for things.



Actually, the bug has to walk a first edge, so I drew all my pictures with the front-bottom edge as its first move.

I seem to be drawing – and getting a feel for - the legitimate paths for the bug's motion. The strategy I think I am falling into is:

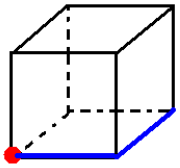
1. Count all the legitimate paths.
2. Count all the possible paths.

Then the probability we seek is the ratio of these two counts.

The second part is easy. Since the bug has three choices at each vertex as to which edge to travel next (including going back over the one it just traversed) there are 3^7 paths the bug can walk in total.

Now back to the first part, counting the paths in which the bug goes to a new vertex at each step.

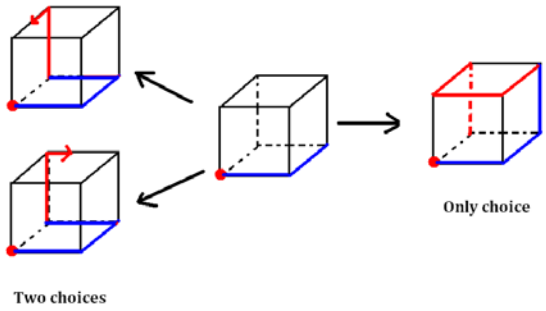
The bug has to walk a first edge, and then a second new edge. Its path starts with the structure:



There are three faces containing the starting vertex, and each face as two pairs of edges that can be a start as shown. So that's six options that look like the above.

How can the bug go next?

If it stays in the same face of the cube for its next move, then there are only two legitimate ways to carry on. If it moves out of this plane, then there is only one. (Surprise!)



So with each of our six options for the initial two moves there are three more options for how to finish the bug walk. This means that there $6 \times 3 = 18$ legitimate paths and the probability we seek is: $\frac{18}{3^7} = \frac{2}{3^5} = \frac{2}{243}$. Done!

Extension 1: Suppose the bug is to walk across the diagonal faces of the cube from vertex to vertex. Starting at a given vertex in how many different ways can the bug take seven steps to visit each vertex exactly once?

Extension 2: Starting at one vertex of an icosahedron and walking from edge to edge of the solid, how many paths of 11 edges visit all 12 vertices?

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