Curriculum Burst 55: A Ratio of Areas

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> Let
> $$S_1 = \{(x, y) \mid \log_{10}(1 + x^2 + y^2) \le 1 + \log_{10}(x + y)\}$$
> and
> $$S_2 = \{(x, y) \mid \log_{10}(2 + x^2 + y^2) \le 2 + \log_{10}(x + y)\}.$$
> What is the ratio of the area of S_2 to the area of S_1?

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

MATHEMATICAL TOPICS

Logarithms. Circles.

COMMON CORE STATE STANDARDS

A-REI.D.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

F-BF-B.5 Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: DO SOMETHING!

SOURCE: This is question # 21 from the 2006 MAA AMC 12A Competition.

THE PROBLEM-SOLVING PROCESS:

As always …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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This question is a visual nightmare! The expressions

$$S_1 = \{(x,y) \mid \log_{10}(1+x^2+y^2) \le 1+\log_{10}(x+y)\}$$
$$S_2 = \{(x,y) \mid \log_{10}(2+x^2+y^2) \le 2+\log_{10}(x+y)\}$$

look really scary. And when I read the actual question, I see it is something about areas. What areas?!!

Deep breath …

Okay … without getting into it, I see that S_1 is a set of points (x,y) that satisfy some equation. Oops! That's not right, they satisfy an inequality.

Do I know what that means?

What if S_1 was something friendlier, like: $S_1 = \{(x,y) \mid 1+x^2+y^2 \le 1+x+y\}$, say, just ignoring the logarithms? That's still too hard. What about simpler still: $S_1 = \{(x,y) \mid x^2+y^2 \le 1\}$ instead? Okay, that's the set of points sitting inside a circle of radius 1.

Alright, I "get" it, in a general sense: S_1 is some set of points sitting in a region of the plane. We can talk about "area of S_1" (assuming I can figure out the shape of the region it represents!).

So, what is S_1? (And S_2 too, but it is probably going to be very similar.)

I really have no choice but to try to do something with:

$$\log_{10}(1+x^2+y^2) \le 1+\log_{10}(x+y).$$

Raise everything to the tenth power?

$$10^{\log_{10}(1+x^2+y^2)} \le 10^{1+\log_{10}(x+y)}$$
$$1+x^2+y^2 \le 10(x+y)$$

Oh … this looks like a circle!

$$x^2-10x+y^2-10y \le -1$$
$$x^2-10x+25+y^2-10y+25 \le 49$$
$$(x-5)^2+(y-5)^2 \le 49$$

The set S_1 is the interior of a circle of radius 7 and so has area 49π.

I can see that

$$\log_{10}(2+x^2+y^2) \le 2+\log_{10}(x+y)$$

gives

$$2+x^2+y^2 \le 100(x+y)$$
$$(x-50)^2+(y-50)^2 \le 4998$$

S_2 is the interior of a circle and has area $\pi\left(\sqrt{4998}\right)^2 = 4998\pi$. The ratio of the areas is:

$$\frac{4998}{49} = \frac{4900+98}{49} = 102.$$

Wow!

Extension: Enter $y = x^{\frac{1}{\log_{10}(x)}}$ in a graphing calculator and have the calculator sketch this graph for you. Then explain what you see!

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