Curriculum Burst 53: Largest Common Divisor

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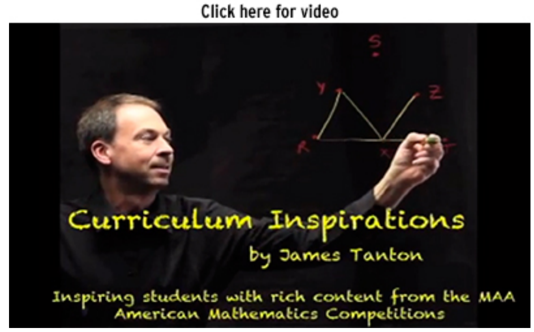
> What is the largest integer that is a divisor of $(n+1)(n+3)(n+5)(n+7)(n+9)$ for all positive even integers n?
>
> (A) 3 (B) 5 (C) 11 (D) 15 (E) 165

QUICK STATS:

Click here for video



MAA AMC GRADE LEVEL

This question is appropriate for the lower high-school grades.

MATHEMATICAL TOPICS

Numbers

COMMON CORE STATE STANDARDS

F-IF.A.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 6: **ELIMINATE INCORRECT CHOICES**

SOURCE: This is question # 18 from the 2003 MAA AMC 10b Competition.

THE PROBLEM-SOLVING PROCESS:

As always, the first step is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I find this question a little confusing. Just to get a feel for it, let me choose some positive even values for n and see what kind of numbers we're talking about. (I do understand that we're looking for a common factor of all the numbers we could see).

$n=2$: The number is $3\times5\times7\times9\times11$.

$n=4$: The number is $5\times7\times9\times11\times13$.

$n=6$: The number is $7\times9\times11\times13\times15$.

And so on.

Okay, we're always talking about the product of five consecutive odd numbers.

All the examples I've picked so far have a common factor of 11. But that is just coincidence: it is clear that $13\times15\times17\times19\times21$, for example, is not divisible by 11. Choice (C) is out.

I can't help but want to focus on choice (E). We have $165=5\times33=3\times5\times11$. It too can't be the right answer as not all examples are divisible by 11 and so not all can be divisible by 165.

Alright, that leaves options (A), (B) and (D).

Looking at choice (A): Must a product of five consecutive odd numbers be divisible by three? It seems that way in the examples above: each product contains a 3, or a 9, or a 15. That is, they each contain an odd multiple of three.

Hmm. Here are all the odd numbers and the multiples of three among them:

1 $\boxed{3}$ 5 7 $\boxed{9}$ 11 13 $\boxed{15}$ 17 19 $\boxed{21}$ 23 …

These multiples of three are spaced three odd numbers apart. This means any string of five consecutive odds is sure to contain at least one of them. Each product in this question is thus sure to be divisible by three. Option (A) is a possible answer. But is 3 the <u>largest</u> common divisor?

By the same reasoning we see that any five consecutive odd numbers is sure to contain a multiple of five. So (B) is also a possible answer.

1 3 $\boxed{5}$ 7 9 11 13 $\boxed{15}$ 17 19 21 23 $\boxed{25}$ …

But any number that is divisible by both 3 and 5 is also divisible by 15. This means that (D) is also a possible answer!

Well, the question asks for the largest common divisor listed, so (D) is in fact the answer to the question.

Extension 1:
Prove that the product of any three consecutive integers is sure to be divisible by 6.

Prove that the product of any four consecutive integers is sure to be divisible by 24.

Prove that the product of any five consecutive integers is sure to be divisible by 120.

Prove that the product of any ten consecutive integers is sure to be divisible by 3628800.

Extension 2: What is the largest common factor of all products of five consecutive even numbers?

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