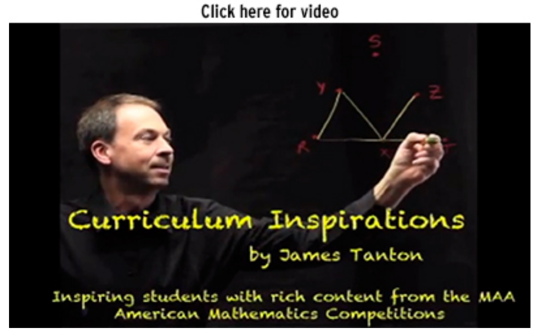
Curriculum Burst 49: Red and White Chips

By James Tanton, PhD, Mathematics, Princeton 1994; MAA Mathematician in Residence

> A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

THE QUICK STATS:

Click here for video



MAA AMC GRADE LEVEL

This question is appropriate for the 10^{th} grade level.

MATHEMATICAL TOPIC

Probability

COMMON CORE STATE STANDARDS

7.SP.C.8 Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.

S-CP.B Use the rules of probability to compute probabilities of compound events in a uniform probability model.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.

PROBLEM SOLVING STRATEGY

ESSAY 9: AVOID HARD WORK

SOURCE

This is question # 23 from the 2001 MAA AMC 10 Competition.

THE PROBLEM-SOLVING PROCESS:

The appropriate start, as always, is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I see right away one approach to this problem: go through all the possible ways to select chips that result in seeing two whites before seeing three reds.

Case 1: Pull white and then another white.

The probability of this is $\frac{2}{5}\times\frac{1}{4}$.

Case 2: Pull white, then red, then white.

The probability of this is $\frac{2}{5}\times\frac{3}{4}\times\frac{1}{3}$.

Case 3: Pull red, then white, then white.

The probability of this is $\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}$.

Case 4: Pull white, then red, then red, then white.

The probability of this is $\frac{2}{5}\times\frac{3}{4}\times\frac{2}{3}\times\frac{1}{2}$.

And so on! But I am left wondering … How many cases are there? And when I get them all, adding up all the fractions to get the final probability we seek is not doubt going to be tedious. There has to be an easier way to "see" what is happening in this problem and avoid this case-by-case analysis. Hmm.

At points like these, when I am stuck on a problem and am hoping for a flash of insight, I usually go for a walk and let my brain mull! (Do you have time for a walk right now?)

Let's see … We want the probability of selecting the two white chips before we see all three red chips. That is, as we play this game, we want the probability that any chips left in the box once we stop playing are all red. (Then we are sure to have pulled out all the whites first.)

Does thinking of it that way help? I am not sure .. it still feels as though we need to break the problem into cases – having three red chips left behind when we stop, two left behind when we stop, one left behind when we stop and zero left behind when we stop.

Ooh! Hang on. Would the game ever go to the point of pulling a fifth chip to leave zero behind? If that fifth chip is white, then we would have already seen three red chips and stopped the game. If that fifth chip is red, then we would have already seen two white chips and stopped the game. We will never pull a fifth chip!

Hmm. That feels significant.

So what if we turned this problem around and imagined playing it this way:

Start by selecting a chip at random to be called "the fifth chip." It will never be used, so remove it from the box. Play the game with the remaining four chips.

If that fifth chip is red, you will be guaranteed to see the two white chips (and not three reds). If that fifth chip is white, you will be certain to see three reds (and not two whites).

Is this the same game? I am a little nervous about this as there might be two or three chips we never pull and so there is more than one candidate for a "fifth chip." Hmm.

How about this then?

Suppose I play the game and after I am done my friend Poindexter happens to come along and keeps pulling chips from the box. Then there is definitely a first chip, a second chip, all the way up to a fifth chip. I ignore Poindexter and still ask, what are the chances that I see two whites before three reds?

Poindexter's actions don't affect the game in any way, but I can still say for sure, if his fifth chip was red, I did see two whites first, and if his fifth chip was white, I did see three reds.

So the question boils down to: What are the chances the fifth chip Poindexter sees is red? Nothing more! And the answer to that is $3/5$. (Seeing that easy answer was hard!)

Optional Exercise : Continue the case-by-case approach outlined at the beginning of the essay and obtain the answer $3/5$.

Extension: Answer the question again but for 101 white chips and 102 red chips in the box.
How about 101 white and 1002 red chips?

Curriculum Inspirations is brought to you by the Mathematical Association of America and the MAA American Mathematics Competitions.

MAA acknowledges with gratitude the generous contributions of the following donors to the Curriculum Inspirations Project:

The TBL and Akamai Foundations
for providing continuing support

The Mary P. Dolciani Halloran Foundation for providing seed funding by supporting the Dolciani Visiting Mathematician Program during fall 2012

MathWorks for its support at the Winner's Circle Level