Curriculum Burst 43: A Big Last Digit

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> When 1999^{2000} is divided by 5, the remainder is … ?

THE QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the 8^{th} grade level.

MATHEMATICAL TOPIC

Number Sense; Exponents

COMMON CORE STATE STANDARDS

8.EE.A Work with radicals and integer exponents.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP8 Look for and express regularity in repeated reasoning.

PROBLEM SOLVING STRATEGY

ESSAY 5: **SOLVE A SMALLER VERSION OF THE SAME PROBLEM**

SOURCE

This is question # 24 from the 1999 MAA AMC 8 Competition.

THE PROBLEM-SOLVING PROCESS:

The best start …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

Obviously I am not going to evaluate the product $1999 \times 1999 \times \cdots \times 1999$ two thousand times and divide the crazy answer by five just to find its remainder. That is ludicrous! Nonetheless, the problem wants me to find that remainder. Hmm.

Just to get a feel for things, let's try a similar problem with smaller numbers:

What is the remainder of 1999^{20} *upon division by* 5*?*

That's still too hard. What about smaller still?

What is the remainder of 1999^{2} *upon division by* 5*?*

I suppose I can do that, but why not smaller still?

What is the remainder of 1999 *upon division by* 5*?*

Well, 1995 is a multiple of five and so 1999 leaves a remainder of 4.

Okay .. that's fine. So working our way back up, what about 1999^2 then?

I really don't like the number 1999. It seems hard to work with. Is there and easier way to think of it? What if I use $1999 = 2000 - 1$? Then:

$$1999^2 = (2000-1)^2 = 4000000 - 2000 - 2000 + 1.$$

	2000	-1
2000	4000000	-2000
-1	-2000	1

It is now clear that 1999^2 is one more than a multiple of five.

How about 1999^3? It is 1999^2 times another 1999.

$$1999^3 = (4000000 - 2000 - 2000 + 1)(2000-1)$$

I don't need to do all the work expanding this. I can see that the answer is going to be:

$$1999^3 = \text{sums of multiples of } 2000 \quad - \; 1 .$$

So 1999^3 is one less than a multiple of five.

Let's keep going! Multiplying 1999^3 by another 1999 gives:

$$\begin{aligned} 1999^4 &= (\text{lots of } 2000s \; -1)(2000-1) \\ &= \text{lots more } 2000\text{ s} \;\; + \; 1. \end{aligned}$$

And again!

$$\begin{aligned} 1999^5 &= (\text{lots of } 2000s \; +1)(2000-1) \\ &= \text{lots more } 2000\text{ s} \;\; - \; 1. \end{aligned}$$

And again!

$$\begin{aligned} 1999^6 &= (\text{lots of } 2000s \; -1)(2000-1) \\ &= \text{lots more } 2000\text{ s} \;\; + \; 1. \end{aligned}$$

We see that 1999 to an odd power is always going to be one less than collection of multiples of 2000, and 1999 to an even power always one more.

So

$$1999^{2000} = \text{lots of } 2000s \quad + \; 1$$

and thus leaves a remainder of 1 upon division by five!

Extension: What remainder does 1999^{2000} leave upon division by 40? Upon division by 16? Upon division by 500? Upon division by 7?

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