Curriculum Burst 32: Mean, Median and Mode

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When the mean, median and mode of the list

$$10, 2, 5, 2, 4, 2, x$$

are arranged in increasing order, they form a non-constant arithmetic progression.
What is the sum of all possible real values of x?

SOURCE: This is question # 14 from the 2000 MAA AMC 12 Competition.

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

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MATHEMATICAL TOPICS

Probability and Statistics: Measures of Central Tendency.

COMMON CORE STATE STANDARDS

S-ID.A: Summarize, represent, and interpret data on a single count or measurement variable.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 6: **ELIMINATE INCORRECT CHOICES**

THE PROBLEM-SOLVING PROCESS:

As always …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

The feature of this question that makes me feel uneasy is the vocabulary: *mean*, *median*, *mode*, and even *arithmetic progression*. I need to get the meaning of these words back in my head.

Mean is "average," and the average of these seven data is:

$$\text{Mean} = \frac{10+2+5+2+4+2+x}{7} = \frac{25+x}{7}.$$

I remember mode is the most frequent data point. Without even knowing the value of x, it is clear that "2" occurs the most times.

$$\text{Mode} = 2.$$

I have a faint memory that median is the middle number of the data set, but you need to first arrange the data in order. The best I can do to order the numbers for now is to write:

$2\ \ 2\ \ 2\ \ 4\ \ 5\ \ 10$, and also x

If x is between 2 and 4, then x will be the middle number. (This is true even if $x = 2$ or if $x = 4$.) If $x < 2$, then 2 would be the middle number, and if $x > 4$, then 4 is at the middle. So we have:

$$\text{Mean} = \begin{cases} x & \text{if } 2 \le x \le 4 \\ 2 & \text{if } x < 2 \\ 4 & \text{if } x > 4 \end{cases}.$$

Umm, what was the question?

The mean, median and mode make a (non-constant) arithmetic progression.

What does that mean? Okay, I remember that a sequence whose entries increase from term to term by a constant amount (such as $4, 7, 10, 13, 16, \ldots$) is an arithmetic progression. (And I guess "non-constant" means sequences like $4, 4, 4, 4, 4, \ldots$ which increase by zero from term to term are not allowed.) We want the mean, median and mode to form a non-constant arithmetic progression. That is, they need to constitute a set of three numbers $a\ b\ c$ with a non-zero "step size" from a to b, and then the same step size from b to c.

Okay … I can handle that. Let's look at our three numbers and see when we can make this happen!

This question naturally splits into three cases.

CASE $x < 2$: Our three numbers are $\frac{25+x}{7}$ and 2 and 2. These can't be in non-constant arithmetic progression! This case is out!

CASE $x > 4$: Our three numbers are $\frac{25+x}{7}$ and 2 and 4. Okay, $\frac{25+x}{7}$ could be 0 or 3 or 6, meaning x could be -25 or -4 or 17. Only $x = 17$ has $x > 4$.

CASE $2 \le x \le 4$: Our three numbers are $\frac{25+x}{7}$ and 2 and x. Hmm. This seems too "loose" to pin down. What can we do? Well, we do have that x is greater than 2, so either $2 < x < \frac{25+x}{7}$ or $2 < \frac{25+x}{7} < x$. And we need the same "step size" for each.

In the first situation this means $x - 2 = \frac{25+x}{7} - x$, giving $7x - 14 = 25 + x - 7x$ and $x = \frac{39}{13} = 3$. The second scenario requires $\frac{25+x}{7} - 2 = x - \frac{25+x}{7}$, yielding $2(25+x) = 7x + 14$ giving $x = \frac{36}{5} > 4$. Not possible.

So only $x = 17$ and $x = 3$ give the desired result, and their sum is 20!

Extension: a) Find five data values with median $= 10$, mode $= 10$, mean $= 1000$.
b) Find five data values with median $= 10$, mode $= 1000$, mean $= 10$.
c) Can you find five data values with median $= 1000$, mode $= 10$, mean $= 10$?
d) Repeat the previous three parts for SIX data values!

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