Curriculum Burst 31: Coloring a Pentagon

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> Each vertex of a convex pentagon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?

SOURCE: This is question # 22 from the 2010 MAA AMC 10a Competition.

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the 10th grade level.

MATHEMATICAL TOPICS

Counting: Permutations and Combinations

COMMON CORE STATE STANDARDS

S-CP.9: Use permutations and combinations to compute probabilities of compound events and solve problems.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

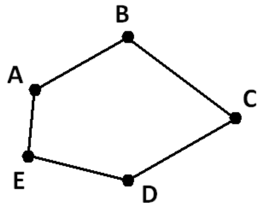
ESSAY 7: PERSEVERANCE IS KEY

THE PROBLEM-SOLVING PROCESS:

As always …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

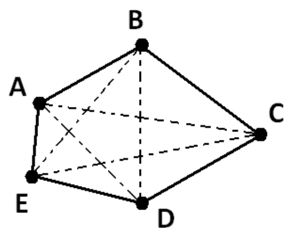
This question seems complicated! We have five points A, B, C, D and E making a (convex) pentagon.



Each vertex is to be colored one of six colors, but in a way such that the endpoints of any diagonal of the pentagon are sure to have different colors.

Umm. What's a "diagonal" of a pentagon?

Let's "nut our way" through this. The diagonal of a square or a rectangle, I know, is a line connecting opposite vertices. Although not symmetrical, I guess the analogous idea for a pentagon would be the five dashed lines here:



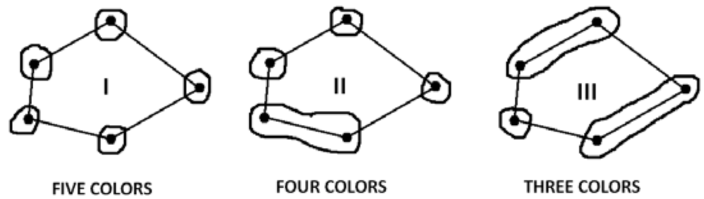
Now I see why we needed a convex pentagon: we want these diagonals to stay inside the figure. (Actually, for this question, does that matter?)

Alright, so B and E can't be the same color, and A and D can't be the same color, and so on. (But neighboring vertices, like A and B, can be the same color. There is nothing in the question stopping that.)

So what types of coloring schemes avoid diagonals having the same color? There's one easy scenario: All five vertices have different colors. Call this SCHEME I.

We could also have two neighboring vertices the same color, and the three remaining vertices different colors. Call this SCHEME II.

Three consecutive vertices the same color is not allowed. (It gives a bad diagonal!) How about two pairs of neighboring vertices, one pair one color, the other pair a second color? The single remaining vertex would have to be a third color. Call this SCHEME III.



A little thought shows these are all the options. (But note, with rotations, there are 5 versions of scheme II and 5 versions of scheme III.) Now we need to count the number of possible colorings for each scheme.

SCHEME 1: Vertex A can be any one of 6 colors, vertex B any one of 5 colors, and so on, to vertex E any one of 2 colors. This gives $6\times5\times4\times3\times2=720$ colorings via scheme I.

SCHEME II: There are 5 options for the location of the pair, 6 choices for the color of the pair, 5 choices for its clockwise neighbor, 4 for its counterclockwise neighbor, and 3 for its opposite vertex. This gives $5\times6\times5\times4\times3=1800$ scheme II colorings.

SCHEME III: There are 5 options for the location of the single vertex and 6 color choices for it. There are 5 color choices for its clockwise neighbor and 4 for its counterclockwise neighbor. That gives $5\times6\times5\times4=600$ scheme III colorings.

That's it, we have a total of $720+1800+600=3120$ coloring schemes in all!

EXTENSION: What if the figure were five beads on a string bracelet, which could be picked up and flipped and rotated? Many of the 3120 patterns we counted would now be considered identical. How many essentially distinct colorings of the beads (from a pool of six colors) are there for a five-bead bracelet such that no two beads on the ends of a "diagonal" are the same color?

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