Curriculum Burst 30: Chords in a Circle

By Dr. James Tanton, MAA Mathematician in Residence

Two points on the circumference of a circle of radius r are selected independently at random. From each point a chord of length r is drawn in a clockwise direction. What is the probability that the two chords intersect?

SOURCE: This is question # 20 from the 2010 MAA AMC 10a Competition.

QUICK STATS:

YouTube Click here for video

MAA AMC GRADE LEVEL

This question is appropriate for the 10th grade level.

MATHEMATICAL TOPICS

Geometric Probability.

COMMON CORE STATE STANDARDS

G-C.2: Identify and describe relationships among inscribed angles, radii, and chords.

S-CP.1: Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGIES

ESSAY 2: **DO SOMETHING!**
ESSAY 4: **DRAW A PICTURE**

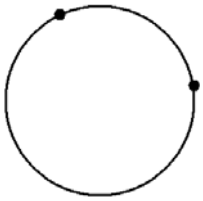
THE PROBLEM-SOLVING PROCESS:

As always …

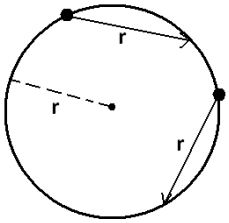
STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

To understand the question I really need to draw a picture.

Two points on the circumference of a circle are chosen at random. (I am skipping over the word "independently." I'll think about what that means later!) I can draw this:

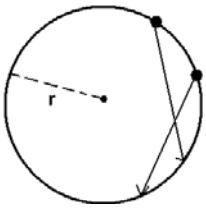


Next, for each point a chord of the same length as the radius of the circle is drawn in a "clockwise direction". This is a little confusing, but it must mean we are doing the following:



I've drawn the center and a radius of the circle here. And I drew arrows on the two chords just to see a "clockwise direction" of some kind. I think I have the correct image.

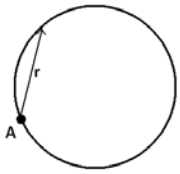
In the picture the two chords do not intersect. If the two points happened to be closer together, the two chords would intersect.



The question is asking for the chances of this second scenario happening. Hmm.

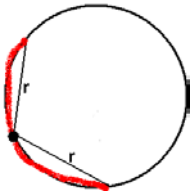
I am always confused in probability questions when objects are tossed simultaneously, or numbers are chosen simultaneously, or points are chosen at the same time. Can we assume that tasks are performed one after another instead? In particular, does our problem "care" if we choose the points in sequence? Not at all!

So let's choose a point, call it A, at random on the circle. (And in a moment we'll choose a second point.) Here it is, and the chord that will go with it.

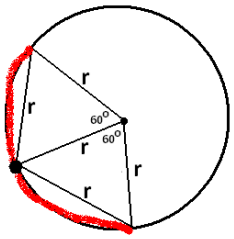


Now let's choose the second point B. Where would B need to land in order to produce intersecting chords? Well, a little experimentation shows B could be anywhere on the shaded arc shown:

What fraction of the circumference of the circle is this arc? This is a bit tricky. There are certainly two chords each of length r associated with this arc.



It seems compelling to draw in three actual radii of the circle.



Oh! And then we see two equilateral triangles, showing that the shaded arc turns through an angle of $2\times 60^\circ = 120^\circ$ along the full circumference of the circle. The shaded arc is one-third of the circumference and the probability that B will land here is $1/3$. This is the answer to the problem!

Extension: What if the points are <u>not</u> chosen independently? For instance, suppose that if A happens to land in the upper-half of the circle, B must be selected from the left half of the circle (and if A lands in the lower semi-circle, there are no restrictions on the placement of B). What now are the chances that their two chords intersect?

Curriculum Inspirations is brought to you by the Mathematical Association of America and MAA American Mathematics Competitions.