Curriculum Burst 22: Walking the Track

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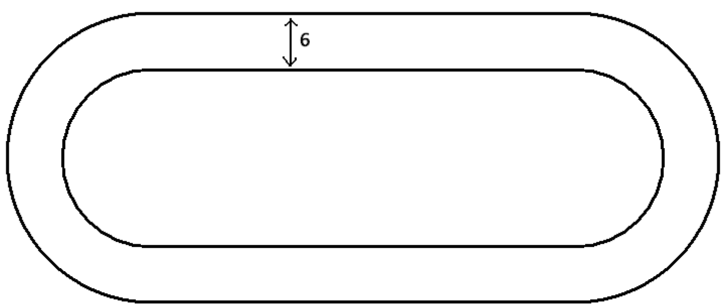
Keiko walks once around a track at exactly the same constant speed every day. The sides of the track are straight, and the ends are semicircles. The track was width 6 meters, and it takes her 36 seconds longer to walk around the outside edge of the track than around the inside edge. What is Keiko's speed in meters per second?



SOURCE: This is question # 12 from the 2011 MAA AMC 10b Competition.

QUICK STATS:

Click here for video

MAA AMC GRADE LEVEL

This question is appropriate for the 10th grade level.

MATHEMATICAL TOPICS

Geometry: The circumference of a circle formula.

COMMON CORE STATE STANDARDS

A-SSE.2: Use the structure of an expression to identify ways to rewrite it.

7.G.4 Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.

MP3 Construct viable arguments and critique the reasoning of others.

MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: **DO SOMETHING**

THE PROBLEM-SOLVING PROCESS:

The best first step is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

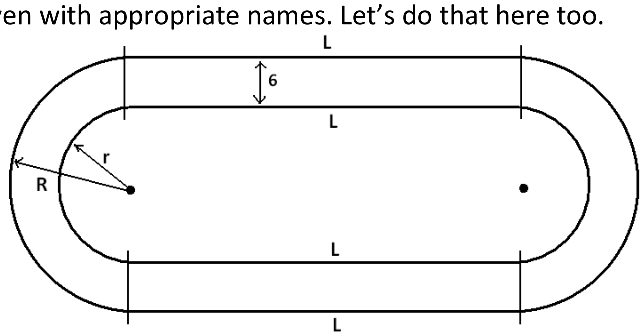
This question feels like a standard geometry problem, except it makes me a little uneasy as key pieces of information seem to be missing: What are the radii of the circular sections? How long are the straight sections?

As I reread the question I realize that I also don't know how long it takes Keiko to walk around the track, be it the inner track or the outer track. The time of 36 seconds is the difference of the two times.

So, I don't know the length of the tracks, and I don't know the time it takes to walk either of the tracks. How am I meant to compute her walking speed?

DO SOMETHING!

In geometry problems we often label the diagrams we are given with appropriate names. Let's do that here too.



Here I've drawn the centers of the circular sections and labeled the radii r and R. I've also called each of the four straight-section lengths L. Noting that each end of the track is composed of semicircles we have:

$$\text{Length of inner track} = 2L + \frac{1}{2}\cdot 2\pi r + \frac{1}{2}\cdot 2\pi r = 2L + 2\pi r$$

$$\text{Length of outer track} = 2L + 2\pi R$$

Since R is larger than r, the outer track is indeed longer than the inner track. So far so good.

I need to reread the question. I've lost track(!) of what we are doing.

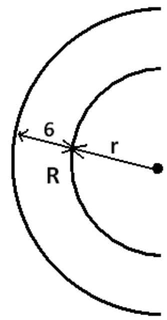
It takes her 36 seconds to longer to walk around the outer track…

Hmm. How much longer is the outer track? The difference of lengths is:

$$\begin{aligned}&(2L+2\pi R)-(2L+2\pi r)\\ &= 2\pi R - 2\pi r\\ &= 2\pi(R-r)\end{aligned}$$

I don't think this is helpful as I don't know the values of the radii! I need to look at the question again.

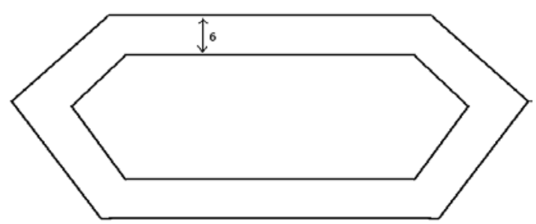
We haven't used that value of 6 anywhere. It must be relevant in some way. Oh … look at this:



The gap between the circular portions of the track is also 6 meters and so we see that $r+6=R$. This means that $2\pi(R-r)=2\pi(6)=12\pi$. We have an actual number!

So 12π is the extra distance she walks in going around the outer track, and it takes Keiko 36 seconds to walk this extra distance. Her speed is thus $\frac{12\pi}{36}=\frac{\pi}{3}$ meters per second. Done!

Extension: What if each end of the track was half a regular hexagon, or half an regular octagon (or half a square?)



See the video www.jamestanton.com/?p=688 for the "wrapping a rope around the Earth" puzzle that connects very nicely with this problem.

Curriculum Inspirations is brought to you by the Mathematical Association of America and MAA American Mathematics Competitions.