Curriculum Burst 14: Town Population

By Dr. James Tanton, MAA Mathematician in Residence

In 1991 the population of a town was a perfect square. Ten years later, after an increase of 150 people, the population was 9 more than a perfect square. Now, in 2011, with an increase of another 150 people, the population is once again a perfect square. Which of the following is closest to the percent growth of the town's population during this twenty-year period?

(A) 42 (B) 47 (C) 52 (D) 57 (E) 62

SOURCE: This is question # 19 from the 2011 MAA AMC 10a Competition.

QUICK STATS:

YouTube
Click here for video

MAA AMC GRADE LEVEL

This question is appropriate for the 10^{th} grade level.

MATHEMATICAL TOPICS

Difference of Two Squares Formula

COMMON CORE STATE STANDARDS

N-Q.2: Define appropriate quantities for the purposes of descriptive modeling.

A-SSE.2: Use the structure of an expression to identify ways to rewrite it.

A-SSE.3: Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

A-APR.4: Prove polynomial identities and use them to describe numerical relationships.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure

PROBLEM SOLVING STRATEGIES

ESSAY 8: SECOND-GUESS THE AUTHOR

THE PROBLEM-SOLVING PROCESS:

As usual, and most important,...

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

Okay. This question is contrived. There is no way that a town's population happens to match perfect squares in this way. Obviously, the author has come up with something clever about the numbers 9, 150 and 150 and the perfect squares.

So… Letting go of the idea that this has anything to do with the real world, I wonder now what clever little tidbit of arithmetic the author has discovered?

DO SOMETHING

The obvious thing to do is to translate the information of the question into mathematics. Let's do it line-by-line:

In 1991 the population of a town was a perfect square.

Call the 1991 population p^2.

Ten years later, after an increase of 150 people, the population was 9 more than a perfect square.

This is a bit confusing. The population in 2001 is "nine more than a perfect square." It must be of the form: q^2+9.

Okay, we have: $p^2+150=q^2+9$

Now, in 2011, with an increase of another 150 people, the population is once again a perfect square.

Another perfect square! That is q^2+9 plus another 150, is a square. We have: $q^2+9+150=r^2$

And now I am lost! What's what?

Let's summarise:

1991 population $=p^2$

2001 population $=p^2+150=q^2+9$

2011 population $q^2+159=r^2$

And what do we want?

REREAD THE QUESTION

We want "*the percent growth of the town's population during this twenty-year period*"

Okay, we need to know the 2011 population and compare it to the 1991 population and express the increase as a percentage. Ugh!

DO SOMETHING

Let's just pick an equation and do something! The last equation looks easiest.

$$q^2+159=r^2$$

Can we rewrite it? (What else can we do?)

$$r^2-q^2=159$$

Ooh! Difference of two squares!

$$(r-q)(r+q)=159.$$

Okay .. The author must have led us to the number 159 for a reason. It's a multiple of three: $159=3\times 53$. And 53 is prime, so these two factors are it. Too perfect! (The question was carefully constructed!) We must have $r-q=3$ and $r+q=53$.

Two equations in two unknowns: let's solve them. (Again, what else is there to do?) Adding gives:

$$\begin{array}{r} r-q=3 \\ r+q=53 \\ \hline 2r\quad=56 \end{array}$$

So $r=28$ with $r^2=(20+8)(20+8)=400+160+160+64=784$.

The 2011 population is 784 and the 1991 population is $784-300=484$ (which happens to be 22^2 !) The percentage increase is $\frac{300}{484}$ which is slightly more than $300/500=60\%$. The answer must be (E)!

Extension: We didn't consider the possibility $(r-q)(r+q)=1\times 159$, the other two-number factorization of 159! Does this lead to another solution? Explore!

Curriculum Inspirations is brought to you by the Mathematical Association of America and MAA American Mathematics Competitions.