Curriculum Burst 12: Non-Congruent Triangles

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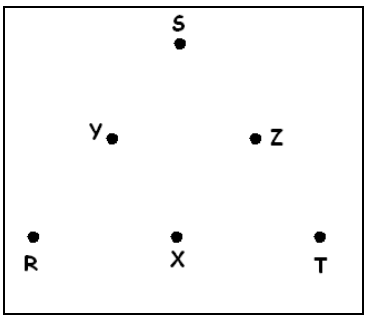
Points R, S and T are vertices of an equilateral triangle, and points X, Y and Z are midpoints of its sides. How many noncongruent triangles can be drawn using any three of these six points as vertices?



(A) 1 (B) 2 (C) 3 (D) 4 (E) 20

SOURCE: This is question # 23 from the 2001 MAA AMC 8 Competition.

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the 8th grade level.

MATHEMATICAL TOPICS

Geometry: Congruence

Click here for video

COMMON CORE STATE STANDARDS

8.G.2: Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure

PROBLEM SOLVING STRATEGIES

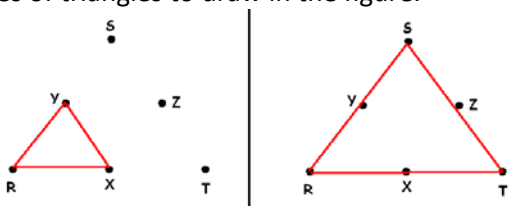
ESSAY 1: **SUCCESSFUL FLAILING**

THE PROBLEM-SOLVING PROCESS:

As always…

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question doesn't feel too scary: all we have to do is draw triangles. And it seems that there are basically only two types of triangles to draw in the figure.



So we are done and the answer is 2 ?

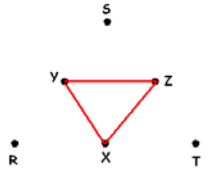
Hmm. That feels too simple.

Maybe I do need to pause and reread the question.

UNDERSTAND THE QUESTION

The tricky words here are *midpoint* and *noncongruent* and *vertices* . Well, *midpoint* just means the middle point of the sides, and *noncongruent* means "not congruent," that is, not the same shape. (Shapes that are reflected and rotated and translated copies of each other are congruent shapes.) So we want triangles that look fundamentally different. (Oh. And *vertex* is just another word for the corner of a shape. So we want triangles with corners on the dots just like we've been drawing.)

Is the trick that there is a middle upside-down triangle? (Maybe some people don't think of this one.)



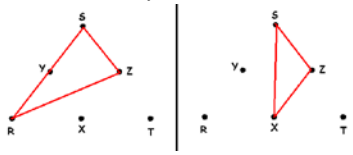
But this is just a congruent copy of the triangle we have up above on the left. It's not a new non-congruent triangle.

Hmm. I am still uneasy about this question.

Reread the question

"noncongruent triangles"

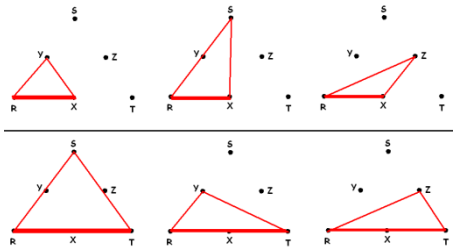
Am I missing any triangles? Oh heavens… yes I am! No-one says we have to draw symmetrical triangles. We can have whacky ones too! For example:



Oh dear. Now I need to get all of those.

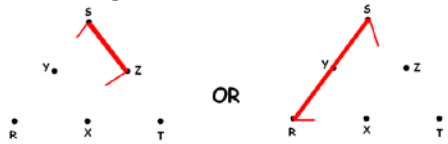
Avoid hard work

I don't want to draw thousands of random triangles and hope I get them all. Can I save some work by being systematic? Let me start with the two examples I have on the left and see what new triangles I can make with the same base for each.



The first four of these pictures give non-congruent triangles. The last two triangles are the "same" as the second one.

What if we used a different base for our triangles? Hmm. It looks like these will give the same basic answers:



Have we missed any triangles? Is there one that doesn't have any length on the outside? I think that's only the upside down triangle XYZ we've already considered.

Okay, that's it. (D): There are 4 noncongruent triangles. Phew!

Extension 1: How many triangles are there in the figure in total? **Extension 2:** Describe the set of motions that take ΔRXZ to each of its congruent copies.

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