Curriculum Burst 9: Differences of Four Numbers

By Dr. James Tanton, MAA Mathematician in Residence

Brian writes down four integers $w > x > y > z$ whose sum is 44. The pairwise positive differences of these numbers are 1,3, 4, 5, 6, and 9. What is the sum of the possible values for w?

SOURCE: This is question # 21 from the 2011 MAA AMC 10b Competition.

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the 10^{th} grade level.

MATHEMATICAL TOPICS

Inequalities

COMMON CORE STATE STANDARDS

A-CED.1: Create equations and inequalities in one variable and use them to solve problems.

A-CED.3: Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure

PROBLEM SOLVING STRATEGIES

ESSAY 6: **ELIMINATE INCORRECT CHOICES**
ESSAY 10: **GO TO EXTREMES**

THE PROBLEM-SOLVING PROCESS:

We always start with…

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

It feels like there is a lot to process in this problem! We have four numbers, whose sum is 44, along with all this information about the differences between the numbers. It's overwhelming! And on top of this, we are being asked something about the possible values of w. How many are there? Do we have to find them all? I feel overwhelmed!

Deep breath.

DO SOMETHING

I don't know what!

GO TO EXTREMES

Okay, we have four numbers $w > x > y > z$, in order from biggest to smallest, with differences between pairs of numbers ranging from 1 to 9. Let's focus on these extreme differences.

The biggest difference of 9 can only occur between the biggest and smallest numbers: w and z. So we can write $w = z + 9$ or $z = w - 9$. Since the question wants us to focus on w, let's write z in terms of it.

We have:

$$w > x > y > w-9.$$

Okay. One less variable to worry about.

The smallest different is 1. This must occur between two neighboring numbers in this list: between w and x, or between x and y, or between y and $w-9$. Let's express these three possibilities as follows, keeping the focus on w as best we can:

CASE 1: $w > w-1 > y > w-9$.
CASE 2: $w > x > x-1 > w-9$.
CASE 3: $w > x > w-8 > w-9$.

ELIMINATE INCORRECT OPTIONS

Case 1 can't be right as it has two numbers, $w-1$ and $w-9$, that differ by 8, and 8 is not listed as one of the allowed differences.

Case 3 cannot be right either: it also has two numbers that differ by 8: w and $w-8$.

Okay, we are in case 2!

$$w > x > x-1 > w-9$$

We still have the differences 3, 4, 5 and 6 to think about.

GO TO EXTREMES

The biggest difference to consider now is 6.

The numbers w and $w-9$ don't differ by 6. And we don't want two adjacent entries to differ by 6 (because then looking one entry further to the left or to the right would give a pair with difference bigger than 6). So there are only two possibilities as to which two entries can differ by 6: x and $w-9$ (so $x = w-3$) or w and $x-1$ (and so $x-1 = w-6$).

CASE 2A: $w > w-3 > w-4 > w-9$.
CASE 2B: $w > w-5 > w-6 > w-9$.

Now what?

REREAD THE QUESTION

In case 2A, the sum of the digits is

$$w + (w-3) + (w-4) + (w-9) = 4w - 16$$

This is meant to equal 44, giving $w = 15$.

In case 2B, the sum of the digits is

$w + (w-5) + (w-6) + (w-9) = 4w - 20$ This is meant to equal 44, giving $w = 16$.

That's it! There are only two possible values for w, and they sum to 31. Wow!

Extension: If one were to pick any six (distinct?) integers, are there sure to be four numbers $w > x > y > z$ whose pair-wise differences match those numbers?

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