Curriculum Burst 6: Areas in Triangles

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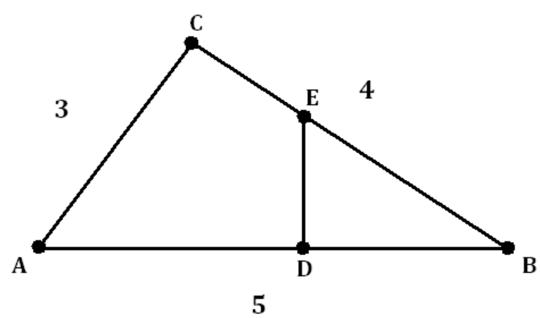
The area of ΔEBD is one third of the area of $3-4-5$ ΔABC. Segment DE perpendicular to segment AB. What is BD?



SOURCE: This is question # 9 from the 2011 MAA AMC 10b Competition.

QUICK STATS:

Click here for video

MAA AMC GRADE LEVEL

This question is appropriate for the 10^{th} grade level.

MATHEMATICAL TOPICS

Geometry: Similar Triangles, Scale.

COMMON CORE STATE STANDARDS

G-SRT.5: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.

PROBLEM SOLVING STRATEGIES

ESSAY 1: [SUCCESSFUL FLAILING: LIST WHAT YOU KNOW]

THE PROBLEM-SOLVING PROCESS:

A vital first step:

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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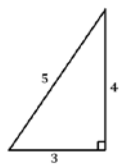
This question has a familiar feel to it. It looks like an exercise from a textbook on geometry (and I have done plenty of geometry textbook questions!). AND it involves a $3-4-5$ triangle, the classic example of a right triangle. Even though I don't see right away what to do, the question doesn't feel too scary.

Let me start by listing what I know about $3-4-5$ triangles.

- A $3-4-5$ triangle is a right triangle with right angle between the sides of lengths 3 and 4.



- We have $3^2 + 4^2 = 5^2$.
- The area of the triangle is $\frac{1}{2} \times 3 \times 4 = 6$.

This means that the area of the small triangle, ΔEBD, is 2.

What else do I know?

This question really does look like an exercise from a geometry book. I have two triangles in the picture so it seems natural then to ask: *Are they similar triangles?*

Well, ΔABC and ΔEBD both share the angle at B. They have at least one angle in common. Actually, the segment DE is perpendicular to the base of ΔABC and so ΔEBD also contains a right angle, like ΔABC. Okay, by the AA principle, ΔABC and ΔEBD are indeed similar.

What do I know about similar triangles?

- All angles match exactly.
- All sides match up to some scale factor k.

But this question is about areas. Do I know anything about similar triangles and area?

- If one scales a figure by a factor k, its area changes as k^2.

Okay! The small triangle has area 2 and the larger, similar triangle has area 6. This tells us that $k^2 = 3$ and so the scale factor between the two triangles is $k = \sqrt{3}$.

Umm. What was the question? What are we being asked to do?

What is BD*?*

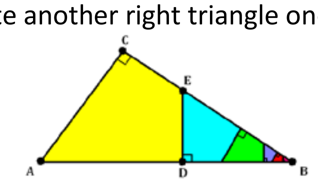
Side BD in the small triangle matches with which side in the large triangle? Well, BD lies between $\angle B$ and the right angle of ΔEBD. The side between $\angle B$ and the right angle in ΔABC is BC, of length 4.

Super! So $BD \times \sqrt{3} = 4$ giving $BD = \frac{4}{\sqrt{3}}$.

We're done!

Comment: In the contest itself, this answer does not appear among the multiple choice options given. Students are expected to recognize this number as $\frac{4\sqrt{3}}{3}$. See the video www.jamestanton.com/?p=513 for a discussion on the strange reasons why school curricula still insist on rationalizing the denominator.

Extension: Suppose we repeat this construction infinitely often: Draw a perpendicular line segment in each right triangle to create another right triangle one-third the area.



Find the areas of each of the colored pieces shown. Their (infinite) sum adds to 6. (Why?) Write down that infinite sum!

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