Curriculum Burst 5: Focus and Directrix

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> A segment though the focus F of a parabola with vertex V is perpendicular to $\overline{FV}$ and intersects the parabola in points A and B. What is $\cos(\angle AVB)$?

SOURCE: This is question # 14 from the 2011 MAA AMC 12b Competition.

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

MATHEMATICAL TOPICS

Geometry: The construction of parabolas.
Trigonometry: Law of Cosines, Double angle formulas.

COMMON CORE STATE STANDARDS

G-GPE.2 Derive the equation of a parabola given a focus and a directrix.

G-SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles.

F-TF.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGIES

ESSAY 1: **SUCCESSFUL FLAILING: LIST WHAT YOU KNOW**
ESSAY 4: **DRAW A PICTURE**
ESSAY 7: **PERSEVERE**

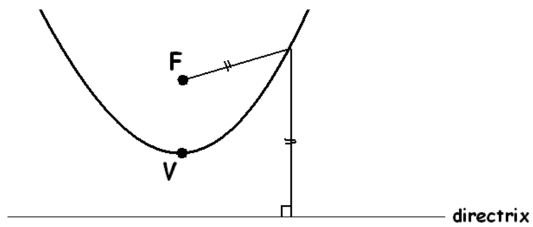
THE PROBLEM-SOLVING PROCESS:

As always …

> **STEP 1:** Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

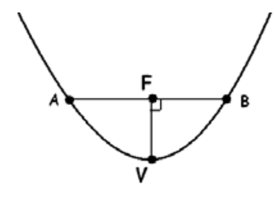
This question gives a lot to digest! I am going to have to draw a picture and attempt to recall all I know about parabolas, their vertices and their directrixes.

A basic picture:

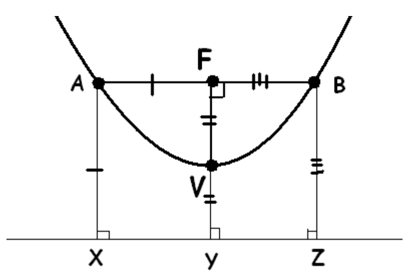


My diagram isn't accurate, but I do remember that a parabola is defined as the set of all points whose (perpendicular) distance to the directrix (a line) equals its distance to the focus (a point). Points on parabolas provide equal distances.

Now for the question we have a segment though F that is perpendicular to the line segment $\overline{FV}$. We can draw that.

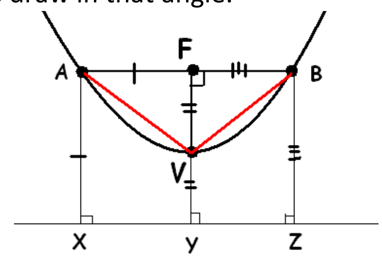


It feels irresistible to draw perpendicular lines to the directrix and note equal distances.

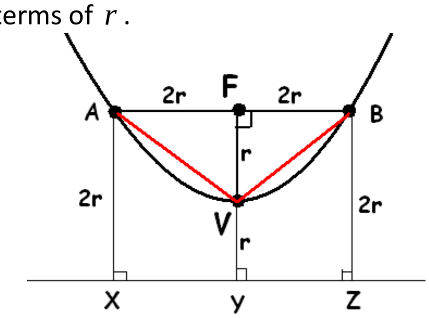


Hmm. $FBZY$ is a rectangle. So is $AFYX$.

Okay … What are we meant to be do? Find the cosine of $\angle AVB$. Let's draw in that angle.



This is confusing. Actually, it is all the little markings of congruent segments that are visually messy. Since we have rectangles, and opposite sides of rectangles are congruent, let's call give length FV a name, say r, and write in all the lengths in terms of r.



(Ooh! Our rectangles are squares!)

We want $\cos(\angle AVB)$. Well… $\angle AVB$ is part of an isosceles triangle with one side of length $4r$ and two legs of length $\sqrt{r^2+(2r)^2}=\sqrt{5}r$. Law of cosines?

$(4r)^2=(\sqrt{5}r)^2+(\sqrt{5}r)^2-2(\sqrt{5}r)(\sqrt{5}r)\cos(\angle AVB)$ This gives

$$\cos(\angle AVB)=\frac{5r^2+5r^2-16r^2}{10r^2}=-\frac{3}{5}.$$

Whoa! We have it. (And a negative answer makes sense since it looks like $\angle AVB$ could be larger than 90° in measure.)

> **Extension:** What is $\cos(\angle FVB)$? How could we use its value to compute $\cos(\angle AVB)$?

COMMENT: The segment $\overline{AB}$ though the focus of a parabola and perpendicular to the segment connecting the focus to the vertex is called the *latus rectum* of the parabola.

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