CLASSROOM CAPSULES

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Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Michael K. Kinyon, Indiana University South Bend, South Bend, IN 46634.

Pizza Combinatorics Revisited

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A decade ago, we corrected the combinatorial mistakes that a nationwide pizza chain made in one of its television advertisements [**1**]. However, the pizza industry has once again failed to master the art of pizza combinatorics in a recent commercial, which introduces a new product consisting of four individually topped pizzas in one box. Each pizza comes with a selection of up to three toppings, out of 17 choices, or one of four specialty pizzas. The commercial asserts that there are more than six million possibilities for the group of four pizzas. Let's determine the accuracy of this assertion.

Not allowing multiple amounts of a topping on a pizza, such as double pepperoni, there are

$$\binom{17}{0}+\binom{17}{1}+\binom{17}{2}+\binom{17}{3}+4=838$$

different possible pizzas. For the four pizzas that are placed in the box, you can have (a) four different pizzas, (b) three different pizzas, (c) two different pizzas, or (d) one kind of pizza. In case (b), once you have three different pizzas, the fourth has to be the same as one of the others. There are three possibilities. In case (c), once you have two different pizzas, the last two can be the same as the first, the same as the second, or one can be the same as the first and the other the same as the second. Again, there are three possibilities. Therefore, keeping in mind that it does not matter how the four pizzas are arranged in the box, there is a total of

$$\binom{838}{4}+3\binom{838}{3}+3\binom{838}{2}+\binom{838}{1}=20{,}695{,}218{,}670$$

possibilities for the group of four pizzas. An easier approach is to consider "order forms" as described in [**1**], also yielding

$$\binom{841}{4}=20{,}695{,}218{,}670$$

possibilities.

Perhaps the commercial is correct since this is *more than* six million!

References

1. Griffin Weber and Glenn Weber, Pizza combinatorics, *College Math. J.* **26** (1995) 141–143.

Using Random Tilings to Derive a Fibonacci Congruence

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It is well known that the nth Fibonacci number F_n, defined by the recurrence $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_1 = F_2 = 1$, is given in closed form by the Binet formula

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

In a series of papers ([**1**]–[**5**]) as well as their recent book [**6**], Benjamin and Quinn (along with others) proved a variety of identities involving very general Fibonacci (and the related Lucas) sequences. They used ingenious counting arguments involving *random tilings* with variously conditioned tiles. A tiling of length n has n *cells* associated with it. For example, a square (1×1 tile) covers one cell and a domino (1×2 tile) covers two. A random tiling involves, at each stage, randomly choosing among the tiles with a specified probability distribution. One then interprets $c_n = F_{n+1}$ as the number of ways to tile the first n cells, for then c_n satisfies the same recurrence relation and initial conditions as those of F_{n+1}. Proving identities involving general Fibonacci and Lucas numbers via random tilings involves viewing an identity "as a story which can be told from two different points of view." (See [**5**, p. 359].)

For example, proving the Fibonacci numbers are given by the Binet formula involves calculating the probability that a random tiling of infinitely many cells using only squares and dominoes is *breakable* at cell n, that is, a square or a domino begins at cell n, in two different ways. The proof considers the probability of choosing a square in our random tiling to be $1/\phi$ and the probability of choosing a domino to be $1/\phi^2$, where ϕ satisfies $(1/\phi) + (1/\phi^2) = 1$, that is, $\phi = (1+\sqrt{5})/2$. This choice of probability distribution implies that the probability of an infinite tiling beginning "with any particular length n sequence of squares and dominoes is $1/\phi^n$" (see [**1**, p. 512]), and hence depends only on n and *not* on the distribution of tiles covering the first n cells. In fact, in [**1**], *every* model has this key feature.

The purpose of this note is to give an example of what happens when the choice of probability distribution for the tiles *does not* exhibit this feature. The price one pays is that instead of an equality one gets a beautiful congruence relation.

Theorem. *Let a and b be positive integers satisfying $a^2 \equiv b \pmod{a+b-1}$. Then, if $\{F_n\}$ is the Fibonacci sequence,*

$$(b+1)a^{n-1}F_n \equiv \left(1-(-b)^n\right) \pmod{a+b-1} \quad \textit{for all} \quad n \in \mathbf{N}.$$