Topology

Erica Flapan, Pomona College (Chair)
Colin Adams, Williams College
Shea Burns, North Carolina AT&T State University
Dick Canary, University of Michigan
Jonathan Rosenberg, University of Maryland

Introduction

Topology serves the undergraduate curriculum in ways that are distinct from other types of mathematics courses. An elementary course in topology has a visual and intuitive appeal that can attract into the major students who may otherwise experience other mathematics courses as "symbol pushing." At an intermediate level, Topology provides a bridge between intuition and formalism, and teaches students how to handle abstraction and construct a rigorous proof. Also, it gives students insight into their analysis courses, helps students understand how to go back and forth between geometric and algebraic reasoning, and exposes students to a diverse range of current mathematical applications from physics, to robotics, to DNA recombination. An advanced undergraduate course in Topology gives students the solid foundation that they will need for the topology, geometry, and analysis they will see if they go to graduate school in mathematics.

Until a few decades ago, a standard undergraduate course in topology consisted of a rigorous development of point set topology that was intended only for advanced mathematics majors headed for graduate school. While point-set topology remains a crucial component of the basic language of mathematics, it is no longer the active area of mathematical research that it was in the first half of the twentieth century. As a result, graduate programs now emphasize geometric and algebraic topology over point set topology. While it is still important to introduce students to fundamental concepts in topology (e.g., continuity in the setting of topological spaces, compactness and connectedness), we recommend that undergraduate courses de-emphasize many of the more esoteric or pathological topics previously taught in traditional point set topology courses (e.g., the order topology or nets). Instead, we recommend that students intending to continue on to graduate school in mathematics be exposed to geometric and algebraic concepts (e.g., the fundamental group and covering spaces).

In addition, it has recently become clear that collaboration and cooperation in problem solving can play an important role in learning mathematics. In addition, preliminary research indicates that collaborative techniques enhance the mathematical experience of some groups of students that are traditionally under-represented in mathematics. We think that courses in topology are particularly well suited to incorporate collaborative problem solving.

More and more mathematics departments have been trying to involve undergraduates in research. Knot theory is an ideal area for undergraduate research projects because of its accessibility and visual appeal. An introductory course in knot theory can open the door to a wide variety of undergraduate research projects. Moreover, the discovery of new applications of topology that began in the 80's has exploded in recent years. Elementary courses in applied topology, knot theory, and combinatorial topology could all draw new students into mathematics.

The inquiry-based topology course described below utilizes collaboration and independent discovery rather than lecturing as the primary pedagogical technique. However, one can easily incorporate collaborative techniques and inquiry into any of the courses we describe by including challenging homework problems that students are expected to work on cooperatively, as well as by including group projects or small group discussions of particular topics.

Thus our study group recommends five different types of Topology courses, depending on the role that a department would like the course to play in their curriculum. While most of these courses could simply be called "Topology," we have given each course a different name in order to make the distinctions between the different courses clearer. We envision that all of these courses would have regular graded homework assignments that would accomplish the cognitive goals of *analytical and critical thinking* and *problem solving*. Ideally, all mathematics departments would offer at least one of these Topology courses on an annual or biennial basis.

Several types of Topology courses: Here are brief [1] descriptions of our recommended courses listed from ***most advanced*** to ***least advanced***.

Honors Topology

Honors Topology is a rigorous Topology course for advanced undergraduate mathematics majors, intended to prepare students for graduate school in mathematics. It covers basic point set topology together with the fundamental group and covering spaces, as well as other advanced topics. The prerequisite for this course is a one-semester course on undergraduate analysis together with a co-requisite of a one-semester course on undergraduate abstract algebra.

Topic goals:

1. Topological spaces and metric spaces
2. Compactness and separation axioms
3. Connectedness and path connectedness
4. Quotient spaces
5. The fundamental group and covering spaces

Inquiry-Based Topology

Inquiry-Based Topology is intended to help mathematics majors think independently and learn to construct and critique rigorous proofs. The course covers point set topology with a focus on compactness and connectedness. Each topic is introduced by a brief motivational lecture by the instructor, after which the students work in groups on worksheets which guide them to discover the material for themselves. Students present the solutions to problems they have solved. The prerequisite for this course is a one-semester course in undergraduate analysis.

In addition to the cognitive goals of analytical and critical thinking and problem solving satisfied by all of our recommended Topology courses, inquiry-based topology accomplishes the additional goals of creativity and curiosity by encouraging students to pose and prove their own conjectures, and the goal of communication skills by requiring students to present their solutions

[1] A more detailed discussion of each of these courses, along with suggested textbooks, can be found in the Appendix: Detailed Course Syllabi.

in class.

Topic goals:

1. Metric spaces
2. Sequential compactness
3. Topological spaces
4. Connectedness and Path Connectedness
5. Compactness

Applied Topology

Applied Topology is intended to expose students with majors in mathematics or the sciences to basic topology and its recent applications. The prerequisite is linear algebra together with an introduction to proofs. However, a more advanced version of this course could be taught with a prerequisite of a one-semester course in undergraduate analysis.

Topic goals:

1. Open sets and topological spaces
2. Continuous functions and homeomorphisms
3. Metric spaces
4. Connectedness and compactness
5. Applications of topology

Knot Theory

Knot Theory is intended to expose students to ideas and proofs in visual mathematics. It is a good course for mathematics majors because it develops students' visual intuition and insight. However, it is also a good course for non-majors, since there are a lot of pictures and students can quickly get into interesting mathematics. It has the potential to draw new students into the major, as well as to give students the background they need to do undergraduate research projects. In addition, this is a good course for future teachers since it complements a traditional geometry course and could give students ideas for topics to bring in high school courses. The prerequisite for this course is the ability to handle mathematical proofs. This could come from a course in Linear Algebra, An Introduction to Proofs, Geometry, or Elementary Number Theory.

Topic goals:

1. Basic definitions, including knots, composition and Reidemeister moves
2. Tabulation of knots
3. Types of knots, including torus knots, satellite knots and braids
4. Surfaces and knots
5. Invariants of knots, including polynomials

Combinatorial Topology

Combinatorial Topology has a profile similar to that of Knot Theory; however, this course might be preferred by a department that would like a survey of several topological topics rather than a course that goes into more depth in Knot Theory.

Topic goals:

1. Topological equivalence
2. Orientability
3. Topology of surfaces
4. Examples of 3-manifolds
5. Basic knot theory

Appendix

Detailed Course Syllabi

Remark*: The presence of a text listed here is not meant to imply an endorsement of that text, nor is the absence of a particular text from the list meant to be an anti-endorsement. The texts are chosen to illustrate the sorts of texts that support the various types of Topology courses described by the authors. Please note that some of the books listed were written by the authors of this report.*

Title of course: Honors Topology
Credit hours/semester: 3 credit course
Target student audience: Junior/senior mathematics majors interested in preparing for graduate study in mathematics or closely related areas, such as theoretical physics.

Course description: This course covers topological spaces and metric spaces, including separation axioms, quotient spaces, compactness, connectedness, path connectedness, and homotopy, the Brouwer fixed point theorem, the fundamental group, covering spaces, and the classification of surfaces (if time permits).

Proposed prerequisites: One semester of undergraduate analysis, plus one semester of abstract algebra. Since the algebra will only be used at the end of the semester for defining fundamental groups and/or homology groups, it can be taken concurrently.

How the course might fit into a program of study. This course would be useful preparation for Ph.D. programs in mathematics. It is intended to be an advanced elective course for mathematics majors.

Course Outline: This course would cover some foundations of general topology, emphasizing the roles of separation axioms, compactness, and connectedness. The notion of homotopy would be explained, as well as why homotopy equivalence is sometimes a more useful equivalence relation than homeomorphism. The course would go on to discuss the fundamental group, simple connectivity, and basics of the theory of covering spaces. The classification of surfaces could be discussed, even if not proved rigorously. A variant of the course might skip covering spaces and instead discuss simplicial homology, getting to the theorem that for a connected polyhedron, the 1^{st} homology is the abelianized fundamental group.

Most topics listed below should take 1-2 weeks to cover. Some topics, especially #s 4, 6, 9, 10, and 12 will each take about 2-3 weeks. The order of some topics may be switched, depending on the choice of textbook.

1. The notion of a topological space. Examples. Equivalent definitions. The case of a metric space.

2. Separation axioms, in particular Hausdorff and normal.

3. Quotient topologies and examples.

4. Compactness and sequential compactness. How compactness interacts with the Hausdorff axiom. The product of compact spaces is compact (finite Tychonoff). (Optional) Full Tychonoff assuming the axiom of choice.

5. Connectedness and path connectedness.

6. (Optional) Urysohn's Lemma and the Tietze extension theorem.

7. (Optional) More on metric spaces, the Arzelà-Ascoli theorem.

8. Homotopy, the fundamental group.

9. The Brouwer fixed point theorem and no retraction theorem.

10. Covering spaces and lifting of maps to covering spaces.

11. (Optional) Classification of surfaces (depending on time, perhaps omit the proof)

12. (Optional) Simplicial complexes and their topological realizations. Computing (combinatorially) the fundamental group of a simplicial complex.

13. (Optional) Simplicial homology as a (computable!) homotopy invariant. Connection between π_1 and H_1.

Possible Textbooks:

1. Armstrong, M. A., *Basic Topology*, Springer, 1983.
2. Crossley, M., *Essential Topology*, Springer, 2005.
3. Kosniowski, C., *A First Course in Algebraic Topology*, Cambridge University Press, 1980.
4. Munkres, J., *Topology*, Prentice Hall, 2000.
5. Runde, V., *A Taste of Topology,* Springer, 2005.
6. Shick, P. L., *Topology: Point-Set and Geometric*, Wiley, 2007.

Modes of delivery: This course is usually taught as a conventional lecture course, with regular problem sets making up a key part of the course. Classes from time to time should be spent discussing problems and having students participate in solving them.

Writing and use of technology: Students would be expected to write clear, rigorous proofs in their problem sets. The instructor might require students to hand in their solutions in LaTeX.

Cognitive Goals: Key cognitive goals of the course are developing facility with constructing and deciphering proofs, learning to "think topologically," and learning to see the beauty and joy of topology.

In addition, the course satisfies the following general principles enumerated by the CUPM:

What should we teach? - Thinking

- This course should help students develop effective thinking skills. The activities of the course are designed to advance and measure students' progress in learning to
    - Read mathematics with understanding
    - Communicate mathematical ideas with clarity and coherence through writing and speaking.
    - Function as an independent mathematical thinker and learner.
- This course should present the beauty and joy of mathematics.

How should we teach? - How people learn

- Students should regularly develop mathematical ideas and explain their ideas both by writing and by speaking.
- The course should emphasize depth of understanding through regular graded homework in addition to class discussion.

Title of course: Inquiry-Based Topology
Credit hours/semester: 3-4 credit hours
Target student audience: Junior and senior mathematics majors.

Course description: The students are guided to discover for themselves the foundational facts about continuity, sequences, sequential compactness, connectedness, the Hausdorff property and compactness in the context of both metric spaces and general topological spaces.

Proposed prerequisite: One semester of undergraduate analysis.

How the course might fit into a program of study: This course is an advanced elective course for mathematics majors. For the majority of students this will be their only course in topology. Students who plan to continue on to graduate school may want to also take more advanced Topology courses, e.g. algebraic topology.

Course Outline:

1. Metric spaces, open and closed sets (1 week)

2. Continuity and sequences in metric spaces, closure and interior (2 weeks)

3. Sequential compactness, the product metric (2 weeks)

4. Abstract topological spaces and continuity (1.5 weeks)

5. Bases, the subspace topology and the product topology (1.5 weeks)

6. Sequences, the Hausdorff property (2 weeks)

7. Connectedness and path connectedness (2 weeks)

8. Compactness (1-3 weeks)

Optional topics:

9. The quotient topology
10. Complete metric spaces

Possible texts: This course is most naturally run without a textbook. However, there are a number of sources for well-prepared materials that can be used to support such a course.

- Sample worksheet sets for this specific course are available from Dick Canary (canary@umich.edu).
- For other possible IBL Topology courses, the *Journal of Inquiry-based learning in Mathematics*, JIBLM, is a source of peer-reviewed, class-tested, inquiry-based notes. (The site contains notes for all sorts of IBL courses, not just topology.)
- More information about inquiry-based learning and resources that support it, contact the *Academy of Inquiry-Based Learning in Mathematics*, AIBL.

Modes of delivery: Each topic is introduced by a brief motivational lecture by the instructor (ideally 5 minutes or less). The students work in groups on worksheets provided by the instructor. The instructor facilitates student discussions with each other but should not direct students to the proof. At the end of each class students present their solutions to any of the problems they have solved. In-class work is supplemented with out-of-class homework, done alternately individually and in groups. Class may begin with student presentations of solutions to homework problems.

Writing and use of technology: One main emphasis of the course is to teach students to write rigorous proofs. It is important that students get detailed feedback on their written work. Students could use power point or beamer for their presentations of homework problems.

Cognitive goals: In addition to teaching the basics of topology, the course is designed to enhance the students' ability to think independently about and discuss serious mathematics. They learn how to construct and critique rigorous mathematical arguments. The students also gain experience presenting mathematical results.

In addition, the course satisfies the following general principles enumerated by the CUPM:

What should we teach? - Thinking

- This course should help students develop effective thinking skills. The activities of the course are designed to advance and measure students' progress in learning to
    - Approach problem-solving with curiosity and creativity, with a willingness to try multiple approaches, persist in the face of difficulties, asses the correctness of solutions, explore examples, pose questions, and devise and test conjectures.
    - Communicate mathematical ideas with clarity and coherence through writing and speaking.
    - Function as an independent mathematical thinker and learner.
- This course should present the beauty and joy of mathematics.

How should we teach? - How people learn

- This course should make mathematics meaningful by using an interactive process in which students participate in the development of new concepts, questions, and answers.
- Students should regularly develop mathematical ideas and explain their ideas both by writing and by speaking.

Title of course: Applied Topology
Credit hours/semester: 3 credit hours
Target student audience: Sophomores, juniors and seniors with majors in either mathematics or the sciences.

Course description: This course is a basic introduction to topology and topological spaces, motivated by applications of topology that have proliferated in recent years. The course covers the standard topics in an introductory point-set Topology course, including topological spaces, open and closed sets, bases, interior, closure, limit points and boundary of subsets, continuous functions, homeomorphisms, connectedness and compactness, as well as a range of applications.

Proposed prerequisites: The prerequisite is linear algebra together with an introduction to proofs. However, a more advanced version of this course could be taught with a prerequisite of a one-semester course in undergraduate analysis.

How the course might fit into a program of study: This course is intended as an elective course in mathematics, appropriate for students intending to do graduate work in mathematics. However, it could also be useful to students intending to pursue advanced degrees in subjects to which the applications are geared.

Course Outline:

1. Preliminaries : This includes a basic introduction to what topology is, perhaps a little history, and then a basic set theory notation, functions, sequences, equivalence relations,

Euclidean n-space and the difference between countable and uncountable sets.

2. Topological spaces: This includes the definition of a topological space, open and closed subsets, and bases for a topological space. Applications can include digital image displays and RNA folding.
3. Interior, closure, limit points and boundary of a set: This includes how the concepts of interior, closure, and limit points relate to one another, and the definition of the boundary of a subset in a topological space. Applications can include Geographic Information Systems.
4. New topological spaces: This includes the subspace topology, the product topology, and the quotient topology. Applications can include surface theory, configuration spaces and robotics.
5. Continuous functions and homeomorphisms: Applications can include robotics.
6. Metric spaces: This includes examples and properties of metric spaces, isometries, and metrizability of topological spaces. The separation axioms are included here as well. Applications can include error-correcting codes and the measure of distance between distinct DNA sequences.
7. Connectedness and path connectedness: This includes the intermediate value theorem and the one-dimensional Brouwer fixed point theorem. Applications can include population modeling and automated guided vehicles.
8. Compactness: This includes properties of compact sets and spaces, compactness in metric spaces, completeness of a compact metric space, applications to the extreme value theorem, limit point compactness, and one-point compactifications.

Optional topics:

9. Dynamics and chaos.
10. Homotopy and degree theory with applications to heartbeat.
11. Fixed point theorems with applications to economic models.
12. Knots with applications to DNA and synthetic chemistry.
13. Graph embeddings with applications to chemistry.
14. Manifold theory and applications to cosmology.

Possible textbooks:

1. Adams, C. and R. Franzoza, *Introduction to Topology: Pure and Applied*, Prentice Hall, 2007.
2. Basener, W., *Topology and its Applications*, Wiley, 2006.
3. Flapan, E., *When Topology Meets Chemistry*, Cambridge University Press and the MAA, 2000.
4. Krantz, S., *Essentials of Topology with Applications*, Chapman and Hall, 2009. (Note this book is at a higher level than the others.)

Another option is to select a more standard topology text supplemented by applications added by the instructor. Possible books include:

5. Munkres, J., *Topology*, 2nd ed., Prentice Hall, 2000.
6. Gamelin, T. and R. Greene, *Introduction to Topology*, Dover, 1999.
7. Mendelson, B., *Introduction to Topology*, Dover, 1990.

Modes of delivery: This course is typically taught in a lecture format with problem sets throughout the course, and several exams.

Cognitive goals: This course takes substantial advantage of applications to motivate student interest. Students should come out of the course with an appreciation for the effectiveness of topology to model a variety of real-world phenomena. Students should learn to think geometrically and topologically, and to understand that mathematics is much more than they may have previously supposed.

In addition, the course satisfies the following general principles enumerated by the CUPM:

What should we teach? - Thinking

- This course should help students develop effective thinking skills. The activities of the course are designed to advance and measure students' progress in learning to
  - State problems carefully, identify essential features of a complex situation, modify problems when necessary to make them tractable, articulate assumptions, appreciate the value of precise definition, reason logically to conclusions, and interpret results intelligently.
  - Approach problem-solving with curiosity and creativity, with a willingness to try multiple approaches, persist in the face of difficulties, asses the correctness of solutions, explore examples, pose questions, and devise and test conjectures.
  - Read mathematics with understanding.
  - Communicate mathematical ideas with clarity and coherence through writing and speaking.
  - Function as an independent mathematical thinker and learner.
- This course should demonstrate the interplay between applications of theory.
- This course should present the beauty and joy of mathematics.

How should we teach? - How people learn

- This course should make mathematics meaningful by using an interactive process in which students participate in the development of new concepts, questions, and answers.
- Students should regularly develop mathematical ideas and explain their ideas both by writing and by speaking.

Title of course: Knot Theory
Credit hours/semester: 3 credit hours
Target student audience: Mathematics and mathematics education majors.

Course description: This course is an introduction to the mathematics behind knots. As an area of mathematical study, knot theory has the advantage that there are lots of pictures, and it is easy to understand and state open problems that very quickly can get into interesting mathematics. It exposes students to an area of mathematics that is highly visual and geometric. Students are often surprised that mathematics includes an area so different from their previous mathematical experience.

Proposed prerequisites: The prerequisite for this course is the ability to handle mathematical proofs. This could come from a course in linear algebra, a transition to proofs, geometry, or elementary number theory.

How the course might fit into a program of study: This course is intended as an elementary elective course in mathematics. It can serve well as a bridge between an introductory proofs course and the more substantial analysis and/or algebra courses. It is a good course for mathematics majors because it develops students' visual intuition and insight. However, it is also a good course for non-majors, since there are a lot of pictures and students can quickly get into interesting mathematics. It has the potential to draw new students into the major, as well as to give students the background they need to do undergraduate research projects. In addition, this is a good course for future teachers since it complements a traditional geometry course and could give students ideas for topics to bring into the high school classroom.

Course Outline:

1. Introduction: A basic introduction to knot theory including some history, composition of knots, Reidemeister moves, links, tricoloration, linking number. Stick number can be introduced as a simple physical invariant that has applications to synthesizing knotted molecules.
2. Tabulating knots: Simple notations for knots, including Dowker notation and Conway notation.
3. Invariants of knots: The traditional invariants of knots, including the crossing number, the bridge number, and the unknotting number of knots.
4. Surfaces and knots: The theory of surfaces with and without boundary. If so desired, one can present the classification of surfaces. Seifert surfaces, the genus of a knot and the proof that genus is additive under composition. Proof that the composition of two nontrivial knots cannot be trivial.
5. Types of knots: Important categories of knots: can include torus knots, satellite knots, hyperbolic knots, 2-bridge knots, alternating knots and almost alternating knots.
6. Polynomials of knots: Introduction to the various polynomials of knots, including the Jones polynomial of a knot using Kauffman's approach through the bracket polynomial and its application to prove that every reduced alternating projection of a knot must have the same crossing number, the Alexander and Homflypt polynomials, and facts about amphicheirality of knots.
7. Biology and chemistry and knots: Applications of knot theory to DNA and DNA knotting, synthesis of knotted molecules.

Optional topics:

8. Knotting and linking in embedded graphs.
9. Knots as a subfield of the more general low dimensional topology.
10. Higher dimensional knotting.
11. Virtual knots.
12. Arc presentations.
13. Finite type invariants.

Possible textbooks:

1. Adams, C., *The Knot Book*, AMS, 2004.
2. Cromwell, P., *Knots and Links*, Cambridge University Press, 2004.
3. Gilbert, N. and T. Porter, *Knots and Surfaces*, Oxford University Press, 1996.
4. Livingston, C., *Knot Theory*, MAA, 1993.
5. Murasugi, K., *Knot theory and its Applications*, Birkhauser, 1996.

Supplementary material can be obtained from:

6. Flapan, E., *When Topology Meets Chemistry*: *A Topological Look at Molecular Chirality*, MAA and Cambridge University Press, 2000.
7. Kauffman, L., *On Knots*, Princeton University Press, 1987.

Modes of delivery: This course is typically taught in a lecture format with problem sets throughout the course, and several exams. In class, faculty can involve students in conjecturing what might be true, and include exercises that ask students to do this.

Writing and use of technology: It is natural to include a writing component in the course. The instructor collects a set of research papers that are relevant to the covered material and that are readable by students with this level of background. Then each student selects one of them, and writes a 5-page paper that explains the research paper at a level understandable to members of the class. The instructor might also use the paper as an opportunity to teach the students to use LaTeX.

Cognitive goals: The primary cognitive goal is to expose students to a very geometrical hands-on version of mathematics. They can use string to play with knots, and to picture various situations. Students will learn to think geometrically and topologically. There are also myriad opportunities to conjecture and to see mathematics as an ongoing field to which students might contribute.

In addition, the course satisfies the following general principles enumerated by the CUPM:

What should we teach? - Thinking

- This course should help students develop effective thinking skills. The activities of the course are designed to advance and measure students' progress in learning to
  - State problems carefully, identify essential features of a complex situation, modify problems when necessary to make them tractable, articulate assumptions, appreciate the value of precise definition, reason logically to conclusions, and interpret results intelligently.
  - Approach problem-solving with curiosity and creativity, with a willingness to try multiple approaches, persist in the face of difficulties, assess the correctness of solutions, explore examples, pose questions, and devise and test conjectures.
  - Read mathematics with understanding.
  - Communicate mathematical ideas with clarity and coherence through writing and speaking.
  - Function as an independent mathematical thinker and learner.
- This course should demonstrate the interplay between applications of theory.

- This course should present the beauty and joy of mathematics.

How should we teach? - How people learn

- This course should make mathematics meaningful by using an interactive process in which students participate in the development of new concepts, questions, and answers.
- Students should regularly develop mathematical ideas and explain their ideas both by writing and by speaking.

Title of course: Combinatorial Topology
Credit hours/semester: 3 credit hours
Target student audience: Mathematics and mathematics education majors.

Course description: This course is an intuitive, example-driven introduction to low dimensional manifolds and knot theory, intended to expose students to ideas and proofs in visual mathematics.

Proposed prerequisites: The prerequisite for this course is the ability to handle mathematical proofs at an intuitive level. This could come from a course in linear algebra, a transition to proofs, geometry, or elementary number theory.

How the course might fit into a program of study: This course could be taught either at an elementary or intermediate level. It could serve as an elective course for both a math majors and math education majors. It is a good course for mathematics majors because it develops students' visual intuition and insight. It is a good course for future teachers since it complements a traditional geometry course and could give students ideas for topics to bring in high school courses. However, it is also a good course for non-majors, since there are a lot of pictures and students can quickly get into interesting mathematics. It has the potential to draw new students into the major, as well as to give mathematics majors the background they need to do undergraduate research projects.

Course Outline: The order of the topics and the length of time devoted to each topic are flexible, depending on the textbook and the interests of the instructor. The following should be considered as a rough guideline.

1. Topological equivalence, including deformations, homeomorphisms, topological properties, using gluing polygons to obtain surfaces (3 weeks).
2. Orientability, including orientation reversing paths, the Möbius strip, Klein bottle, and projective plane (1 week).
3. Topology of surfaces, including the classification of closed surfaces, genus, and Euler characteristic (3 weeks).
4. Examples of 3-manifolds, including the 3-dimensional torus, the 3-dimensional Klein bottle, and the 3-dimensional projective plane, using connected sums and products to obtain more 3-manifolds (3 weeks).
5. Basic knot theory, including equivalence of knots, Reidemeister moves, tricolorability,

linking number, and Kauffman's approach to the Jones polynomial (4 weeks).

Possible textbooks:

1. Arnold, B., *Intuitive Concepts in Elementary Topology*, Dover, 2011.
2. Boltyanskii, V., V. Efremovich, *Intuitive Combinatorial Topology*, Springer, 2001.
3. Carlson, S., *Topology of Surfaces, Knots, and Manifolds*, Wiley, 2001.
4. Farmer, D., *Knots and Surfaces: A guide to discovering mathematics*, AMS, 1995.
5. Huggett, S., *A Topological Aperitif*, Springer, 2009.
6. Messer, R., *Topology Now!*, MAA, 2006.
7. Weeks, J., *The Shape of Space*, Marcel Dekker, 2002.

Topological/geometric games that could be incorporated into this class can be found on Jeff Weeks' website on Topology and Geometry Software .

Modes of delivery: This course would best be taught using a combination of lecture and group work during class. Students would also hand in regular problem sets. There could be one or two midterms together with a final exam.

Writing and use of technology: Students should be required to explore an additional topic on their own and write a term paper and/or give an oral presentation on it. Students would be expected to use power point or beamer for their presentations. The instructor might also use the term paper as an opportunity to teach the students to use LaTeX.

Cognitive goals: This course will expose students to ideas and proofs in visual mathematics.

In addition, the course satisfies the following general principles enumerated by the CUPM:

What should we teach? - Thinking

- This course should help students develop effective thinking skills. The activities of the course are designed to advance and measure students' progress in learning to
  - Approach problem-solving with curiosity and creativity, with a willingness to try multiple approaches, persist in the face of difficulties, assess the correctness of solutions, explore examples, pose questions, and devise and test conjectures.
  - Read mathematics with understanding.
  - Communicate mathematical ideas with clarity and coherence through writing and speaking.
  - Function as an independent mathematical thinker and learner.
- This course should present the beauty and joy of mathematics.

How should we teach? - How people learn

- This course should make mathematics meaningful by using an interactive process in which students participate in the development of new concepts, questions, and answers.
- Students should regularly develop mathematical ideas and explain their ideas both by writing and by speaking.