The Role of Calculus in the Transition from High School to College Mathematics

Report of the workshop held
at the MAA Carriage House
Washington, DC, March 17–19, 2016

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Edited by

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Introduction: Summary Report of the Workshop on The Role of Calculus in the Transition from High School to College Mathematics, Washington, DC, March 17–19, 2016

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In March of 2016, a group of high school teachers, mathematicians, mathematics and science education researchers, state and district supervisors of mathematics, and representatives of organizations with a stake in the issues surrounding calculus in high school, which included the College Board and the National Academy of Sciences, met for three days in Washington, DC to clarify what we know and what we need to know about the role of calculus in the transition from high school to college mathematics. This is a summary of the issues they identified.

Background

In 2015–16, at least 800,000 U.S. high school students were enrolled in a calculus class.[1] This was more than three times the number of U.S. students who took their first calculus class in college.[2] High school calculus enrollments are still growing, with increasing pressure on many students to take calculus ever earlier. In 2016, more than 130,000 students took the AP Calculus[3] exam by the end of grade 11, more than 13,000 by the end of grade 10 (Figure 1).[4] Figure 2 shows how closely the growth of AP Calculus exams taken before grade 12 over the period 2001–16 tracks the growth in the number of AP Calculus exams over the period 1981–96.

If we make the assumption that most of the students who enroll in calculus in high school will go on to matriculate as full-time students in a four-year undergraduate program, of whom there are 1.5 million each year (Eagen et al, 2016), at least half of these full-time, first-year students enter having already studied calculus. Calculus in high school is now commonly perceived to be a prerequisite for admission to the most selective colleges and universities. As a result, high schools increasingly accelerate students so that they can be on track for calculus by grade 12, with the strongest students attempting to complete it by grade 11.

1 Based on the National Center for Education Statistics (NCES) longitudinal study (HSLS:09) reporting that 19% of the 4.4 million students who were in 9th grade in 2009 had taken a calculus course in high school by the time they graduated. Given the 12% growth in AP Calculus exams from 2013 to 2016, this now could be a significant underestimate.

2 Based on data from the Conference Board of the Mathematical Sciences (CBMS), 2013.

3 *Advanced Placement* and *AP* are registered trademarks of the College Board. For simplicity, these designations appear in this publication without the registered trademark symbol ®.

4 Data from the College Board's Advanced Placement Program Summary Reports.

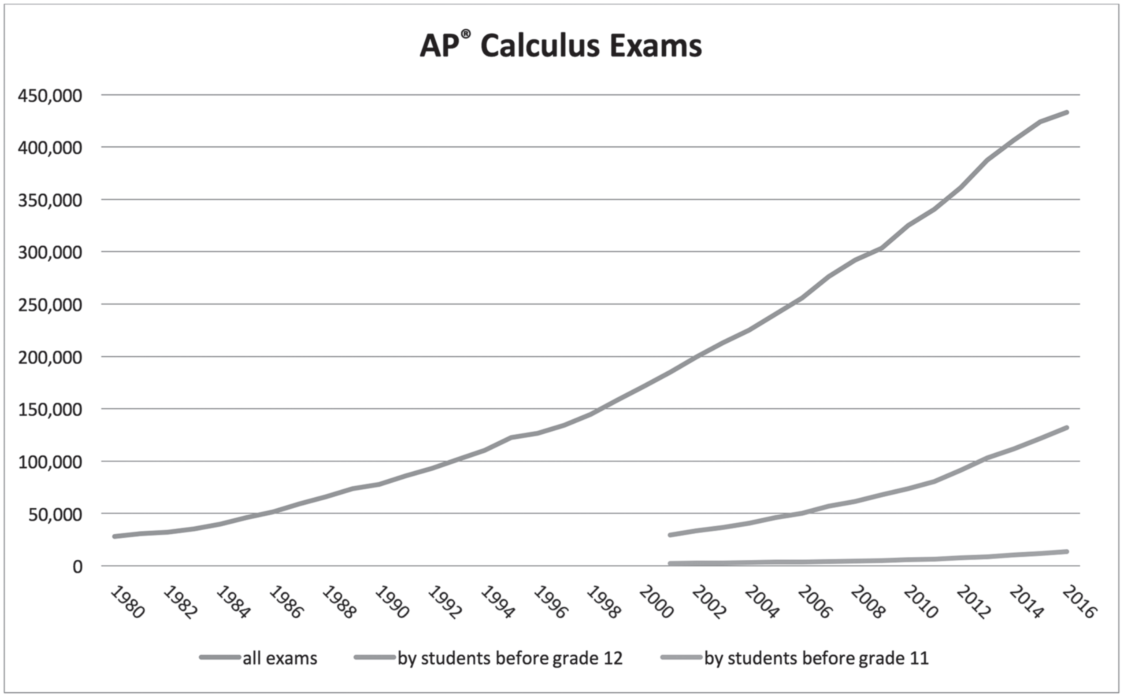

Figure 1. Number of AP Calculus exams taken each year and number of AP Calculus exams taken each year by students before grade 12 and before grade 11.

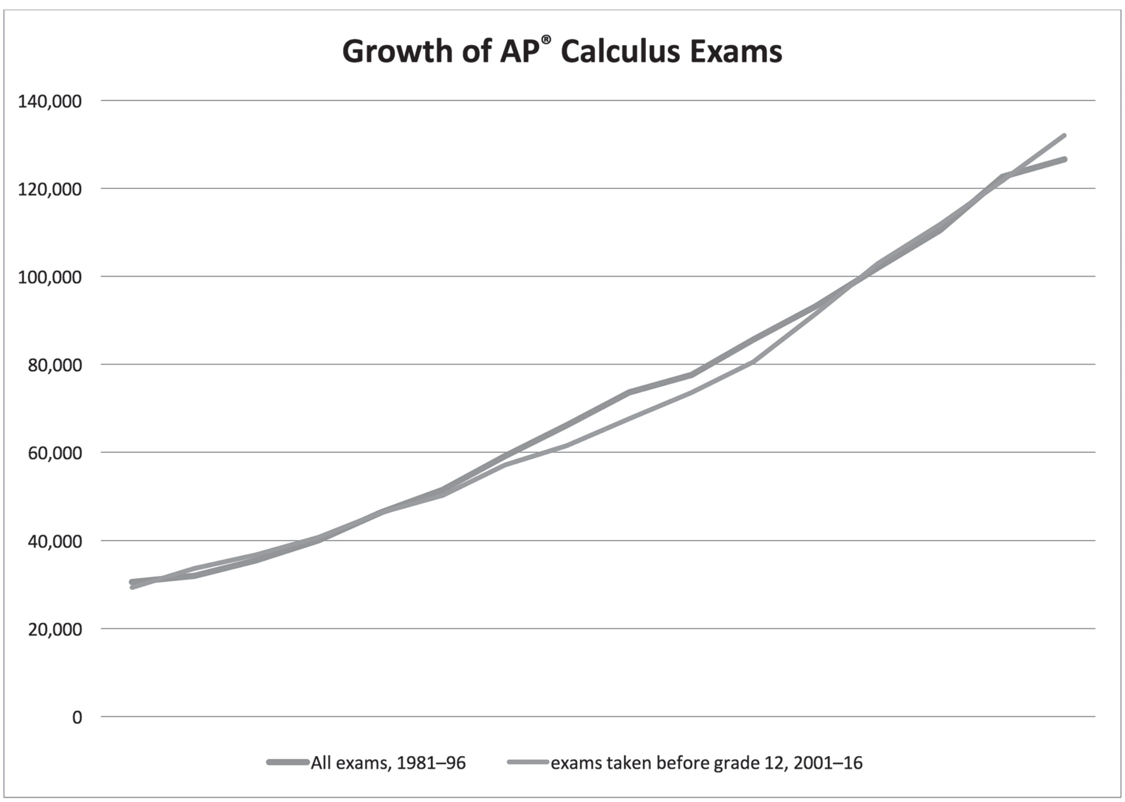

Figure 2. Comparison of growth of all AP Calculus exams starting in 1981 with AP Calculus exams taken before grade 12, starting in 2001.

Calculus teachers find themselves under increasing pressure from parents and administrators to admit into their classes students they know are not adequately prepared.

On the college side of the transition, many of the students who have completed calculus in high school take precalculus, college algebra, or even remedial mathematics when they get to college.[5] This implies lost time and is often discouraging to those who want to pursue a math intensive major. Even for those who go directly into mainstream calculus, differences in expectations, pedagogical approaches, and pacing can make this a difficult transition.

The result is a system that is widely recognized to be problematic. The high school curriculum does not appear to be meeting the needs of the students who have been accelerated. The offerings of our colleges and universities, meanwhile, do not appear to be appropriate. Finally, the obstacles this system erects across the paths of qualified students are not apparent and neither are the full effects on students from underrepresented groups, many of whom do not have access to quality accelerated programs.

What we need to know: Key questions identified at the workshop

Calculus may not be an appropriate goal of the high school curriculum for all students, especially for those students who will not require it for their post-secondary plans. Participants also questioned both what is taught and how it is taught on both sides of the transition. To address these uncertainties, we need far more information about the current situation and its effects.

First, high school calculus is not monolithic. It includes AP Calculus with or without an AP Calculus exam, International Baccalaureate programs, dual enrollment programs, courses taken at local colleges, and online courses, all of these taught by teachers of varying levels of expertise. University-level calculus is also not monolithic, including as it does two-year colleges, liberal arts colleges, public universities, and elite universities, small classes and large lectures, instruction by graduate students and by experienced faculty, those that are experimenting with active learning approaches and those that are wedded to traditional pedagogies. The ground across which these questions must be answered is vast and varied.

Despite these variations, a number of overarching questions were identified in the workshop. These questions reveal how little we know about the role of calculus in the transition from high school to college mathematics.

Across all of these questions, the particular challenges for students from under-resourced schools, students under stereotype threat, and students from families without a history of post-secondary education are of paramount importance.

1. Who takes calculus in high school? When do they take it? Why do they take it? What are the results? This includes both short-term results including what they have learned and long-term results including how it shapes preparation for additional courses in mathematics and future career decisions.
2. Who should be taking calculus in high school? How do we ensure that quality courses are available to all students who should be taking them?
3. Should high school calculus be widely available? Is either calculus as an intellectual achievement or as a sign of intellectual ability sufficient rationale for directing so many students toward it? To what extent does calculus on a high school transcript affect acceptance into college? To what extent does calculus in high school foster increased interest in the mathematical sciences or the mathematically intensive disciplines?
4. How do we know when a student is ready for acceleration into a sequence that will include high school calculus? How should we prepare students for doing this?
5. What are the effects of such acceleration at ever-earlier grades? Are there specific policies and practices that can counter inappropriate acceleration?

5 From NELS:88, 31.5% of the students in the class of 1992 who took calculus in high school also took precalculus in college. From NCES (2013), Table 2-B, 13.5% of the students who took calculus in high school also took a remedial mathematics class when they got to college.

6. How do we build alternate pathways that enable students to back off the track to high school calculus without damaging their prospects for post-secondary studies that are mathematically intensive?
7. How do we ensure that a high school curriculum that includes calculus provides both a sufficient breadth of understanding of the nature of mathematics and ability in the use of the tools of advanced mathematics?
8. What are the obstacles and difficulties that students encounter as they attempt the transition from high school calculus to post-secondary mathematics? What policies and practices are known to be effective at removing obstacles and overcoming difficulties?
9. How can we do a better job of placing students in the appropriate courses when they get to college, and how can we ensure that these courses enable students to succeed in the courses that build upon them?
10. The transition from high school to post-secondary mathematics is frequently detrimental to students' sense of self-efficacy and mathematical identity, especially for women.[6] What are the core issues here, and how can they be addressed?
11. What is the relationship between calculus as currently taught at the post-secondary level and the true needs of those mathematically intensive disciplines that require it?
12. How should colleges and universities respond to the growing proportion of students taking calculus in high school in shaping what and how they teach?
13. How do we ensure that both high school and college instructors are using the most effective methods for teaching calculus?
14. How do we ensure that students have ample opportunity to develop their abilities in critical mathematical practices across both sides of the transition from high school to college mathematics?[7]

What we know: A summary of the background papers

The papers that follow provide an introduction to what we know about the effects of expecting college-bound students to study calculus in high school.

Champion and Mesa (this volume) have mined the National Center for Education Statistics' High School Longitudinal Study of 2009 (HSLS:09) to identify factors that predict which students will enroll in calculus while in high school. They found five factors that account for 86% of the variability in who enrolls in calculus while in high school. In order of importance, they are

1. **Course taken in grade 9:** 41% of those who have taken Algebra 1 before 9th grade will enroll in calculus, as opposed to only 5% of those who take Algebra I in 9th grade and 2% of those who do not see it until after grade 9.
2. **Knowledge of mathematics by grade 9:** 50% of those in the top quartile of knowledge of mathematics as measured by the exam administered for HSLS:09 took calculus and only 2% from the bottom quartile.
3. **Race:** For Asian-American students, 47% took calculus, 19% for White students, and 8% for Black students.
4. **Socioeconomic status:** 38% of those in the top quartile versus 7% in the bottom quartile.
5. **Sense of self-efficacy in mathematics:** 32% from the top quartile, 9% from the bottom quartile.

Sadler and Sonnert (this volume) also show that self-efficacy is only indirectly a product of mathematical competence and performance. High performance requires recognition by parents, peers, relatives, or teachers if it is to raise mathematical identity.

Rosenstein and Ahluwalia (this volume) surveyed 332 Rutgers students who had taken an AP Calculus exam to determine why they chose to take calculus while in high school. Across all scores, about 80% said

6 Sadler and Sonnert, 2015; Ellis, Fosdick, and Rasmussen, 2016

7 Burrill (this volume) summarizes these as reasoning with definitions and theorems, connecting concepts, implementing algebraic and computational processes, connecting multiple representations, building notational fluency, and communicating.

they took the course because it looks good on college applications. For the weaker students, those who scored 3 or less on the AP Calculus exam, this and pressure from teachers, counselors, and friends were the dominant reasons for taking this course. For the stronger students, those who earned a 4 or higher on AP Calculus exam, the dominant reasons, reaching as high as 95% agreement, were "I really liked math when I was in high school," "I enjoy challenging math courses," and "I wanted to learn more higher level mathematics." The authors have no data from those who did not take an AP Calculus exam, but it is likely that their rationales closely resembled those of the students who scored 3 or less.

Teague (this volume) points to the problems inherent in trying to teach a very technical course to those who do not have a strong motivation for being in it:

> Before a student can learn calculus in a manner that has some significant residual, they must want to learn calculus. [...] When the goal is not to develop a deep and abiding understanding and facility with the tools of calculus, but to pass the course with a good grade, either because the students do not value calculus as an important part of their career path or because they know they will be repeating calculus in college, the learning can be quite superficial.

It is therefore not surprising that many of the students who enroll in calculus in high school are not adequately prepared for calculus when they get to college.

Extrapolating from available evidence (see Note on Sources), the breakdown of the first college mathematics class for the 800,000 students who took calculus in high school is approximately as follows (Figure 3): about 150,000 will receive credit for calculus when they get to college and use that credit to enroll in Calculus II or higher. Roughly 250,000 will retake Calculus I; of these, 60% or 150,000 will earn an A or B, but 40% or 100,000 will receive a grade of C or lower. Around 250,000 will need or choose to take precalculus, college algebra, or even remedial mathematics as their first college course. This leaves about 150,000 students who start with a non-mainstream calculus course such as Business Calculus or Statistics, or who take no mathematics when they get to college.

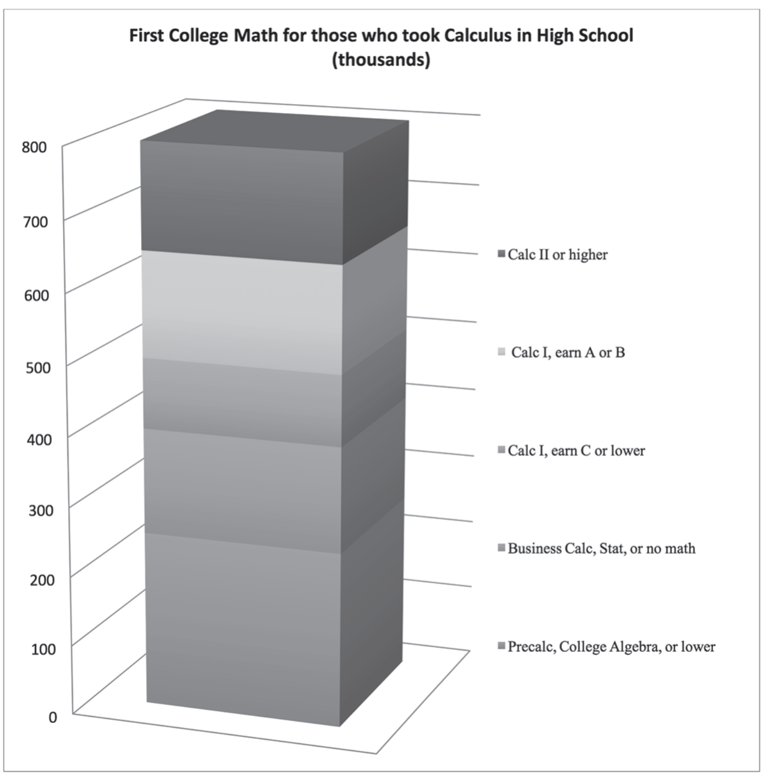

Figure 3. Approximate distribution of the first college mathematics course taken by those who completed calculus while in high school, measured in thousands. For sources of these approximations, see the endnote.

For those students who use Advanced Placement to start at Calculus II or higher, Hedrick and Leonard (this volume) report that they do at least as well as their peers who took Calculus I at the same college or university. Those who do well on an AP Calculus exam are significantly more likely to return for a second year, take more mathematics courses, and pursue a mathematically intensive program. While this supports the validity of the AP Calculus exams, these studies have been restricted to students who earn at least a 3 on one of these exams, about the top third of the students who enroll in calculus while in high school

This raises the question of who *should* take calculus while in high school. Sadler and Sonnert (this volume) have found that grades in high school mathematics for courses up to and including precalculus, combined with SAT or ACT mathematics scores, provide a good predictor of the grade in college Calculus I. They were thus able to control for ability when measuring the effect of taking a high school calculus class. They found that doing so improves the grade of almost all students, a boost of as much as half a grade. This effect tails off at both ends of the spectrum until the benefit completely disappears around two standard deviations above or below the mean.

As they conclude:

> Even students with relatively weak preparation in mathematics appear to benefit from taking a calculus course in high school. While they may not learn all that much calculus (or earn a high grade), the course can bolster their understanding of concepts and build skills that will be used later in college calculus.

This means that access to calculus is important. The data reported by the U.S. Department of Education Office of Civil Rights (2014) that half of all U.S. high schools do not offer calculus should therefore be of concern.

But access to calculus is only part of what will be required to ensure that all students are able to reach their potential. Burrill (this volume) describes some of the fundamental differences between the cultures of high school and college mathematics courses, differences that often trip up students who come to college expecting to continue what they experienced in high school. While the AP Calculus curriculum closely follows that of college Calculus I, expectations, especially at the level of practice standards, can be very different.

Star described a study that illustrates the difference between simply learning procedures and the mathematical practice of reasoning about what has been learned (Maciejewski and Star, 2016). The authors observed that many students, when asked to find a derivative, fail to first simplify the expression, a standard procedure for experts facing the same task. For example, they found that most students when asked to differentiate $(x^3 - 1)/x$ resort to the quotient rule. An expert first simplifies this to $x^2 - x^{-1}$ so that differentiation only requires the exponent rule. They showed that this kind of procedural fluency can be developed by requiring students to try different approaches and reflect on what worked best.

Sadler and Sonnert (this volume) explore the effect of time spent studying on performance in Calculus I. They found a negative correlation between time spent reading the textbook and performance. For the total time spent studying, the highest course taken and performance in that class was a significant factor. For high performing students, increased time spent studying was correlated with improved performance. For low performing students, increased time spent studying led to decreased performance. The authors suggest that for at-risk students, "These students might need specialized guidance—perhaps extra time going over mathematical concepts and/or effective study methods—to enable them to earn higher math grades in high school and calculus grades in college."

Sadler and Sonnert also measure the effect of taking precalculus in college on subsequent performance in Calculus I. They compared the performance of students just below the cut-off who were allowed to proceed directly to Calculus I with those who were just above the cut-off. If precalculus is of benefit, those just below the cut-off should do better in Calculus I than those who are just above. Because of the size of their study, they were able to do this across the range of student levels of preparation for calculus as measured by high school grades in mathematics and SAT or ACT scores. They found that for students whose level of preparation is below the mean, precalculus appears to create a small improvement in calculus scores, but one that is that not statistically significant. For student whose level of preparation is above the mean, taking precalculus decreases grades in calculus by a large and statistically significant amount. For students whose preparation score is half a standard deviation above the mean, the harm incurred was an entire grade level, from B to C.

Finally, we include a piece from Bob Orlin's Math with Bad Pictures that effectively illustrates the situation we face with regard to many of the students who take calculus while in high school. They are not there because they love mathematics or want to prepare for a career in a mathematically intensive field. They are taking the course as a means to establish their credentials as students who will succeed in college. This brings us back to Teague's article and the many questions raised at the workshop about what this implies for how calculus should be taught at both levels.

A note on sources

The numbers represented in Figure 3 are extrapolations from an assortment of hard numbers and percentages from studies that go back as far as 1996. In a few years, data from the NCES High School Longitudinal Study that started in 2009 (HSLS:09) should give us a much more accurate picture of what happened to the high school class of 2012.

The figure of 800,000 students who took high school calculus in 2015–16 is a conservative estimate based on the HSLS:09 report that 19% of the 4.4 million students who entered 9th grade in 2009 subsequently graduated from high school with calculus on their transcripts. Since 2013, the number of students taking an AP Calculus exam has grown by 12%, suggesting that the true number may be closer to 900,000.

In 2016, 220,000 students earned a 4 or 5 on an AP Calculus exam. There is considerable variation in what colleges and universities accept for credit, but almost all of them accept a 4 or 5. How many of these students actually use advanced placement to start with Calculus II or higher is uncertain, but a national study published in 2002 (Christman Morgan) suggests that it is about 75%, a fraction that is consistent with what Rosenstein and Ahluwalia report in this volume. There are colleges and universities that award credit for a 3 on an AP Calculus exam, but the number earning a 3 is relatively small. While there are students who earn college credit for calculus by other routes including International Baccalaureate (IB) and dual enrollment, the 2010 CBMS survey results suggest that only a few tens of thousands of students follow this route. The students who actually earn college credit in this way and use it to go directly to Calculus II is probably less than 20,000. While the true number of students starting at Calculus II or higher may thus be over 150,000, it is certainly below 200,000.

The estimate for the number of high school calculus students who enroll in mainstream Calculus I in college and the breakdown of their grades in this course is based on the data from the MAA's national survey of college calculus, *Characteristics of Successful Programs in College Calculus,* undertaken in 2010. The description "C or below" includes C, D, F, or withdrew from the course. While C is a passing grade, it is usually taken as an indicator that the student is probably not adequately prepared for further courses in the sequence. It should be expected that a student who has passed calculus in high school would do better than C when retaking this course in college.

The number of students who start college mathematics at or below precalculus is based on data from NELS:88 that found that 31% of students with calculus on their high school transcript also had precalculus or lower on their college transcript. This is the oldest piece of data used to construct Figure 3, so its relevance is uncertain. But the rush to enroll ever more students in high school calculus may have the effect of increasing the fraction who arrive at college or university inadequately prepared for Calculus I. The 31% estimate is bolstered by the NCES data from 2013 reporting that of the high school graduates of 2003 who studied calculus in high school, 13.5% enrolled in remedial mathematics when they got to college.

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Factors Affecting Calculus Completion among U.S. High School Students

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Abstract

In this paper we present findings from a preliminary analysis of the transcript data in the High School Longitudinal Study (HSLS:09), with a focus on factors associated with the likelihood of high school students completing calculus in high school. Using proportional flow diagrams of course taking patterns and logistic regression models of the likelihood of students earning credit for calculus in high school, we illustrate differences in calculus completion associated with non-malleable student characteristics such as race, sex, and socioeconomic status (SES), as well as malleable student characteristics, such as knowledge of mathematics in 9th grade, the level of mathematics course they take in 9th grade, and self-efficacy. Confirming and extending findings from prior literature, we conclude that "tracks" through high school mathematics curriculum, together with students' race, socioeconomic status, and self-efficacy converge as effective predictors of whether high school students will complete calculus in high school.

(Acknowledgment: Thanks to Anne Cawley and Ashley Jackson and to the Teaching Mathematics in Community College Research group for feedback on this work.)

Numerous reports highlight the dire state of the country in terms of the preparation of American students in science, technology, engineering, or mathematics (STEM) fields and point to the need to train more students, especially those who are traditionally underrepresented in those fields, in order to diversify the workforce and maintain the country's competitive edge (President's Council of Advisors on Science and Technology, 2012). Likewise, numerous reports, since the early 90s, have highlighted that calculus is a major stumbling block for college students who aspire to earn a STEM degree, especially for those who are underrepresented in these fields (Steen, 1988; Treisman, 1992). The recent National Study of Calculus I found that a large number of students who entered the course with intentions of taking a second course in calculus changed those plans (Rasmussen and Ellis, 2013). Not only that, at the end of one semester of college calculus students' motivation and interest in mathematics significantly decreased (Bressoud, Mesa, and Rasmussen, 2015).

Naturally, what students do prior to enrolling in college has tremendous impact on whether students will pursue, persist, and earn a STEM degree. In their invitation to this workshop, Bressoud and Braddy stated that "high school calculus enrollments are still growing at roughly 6% per year, with increasing pressure on the most advantaged students to take calculus ever earlier" with over 100,000 students taking an AP Calculus exam by the end of grade 11. These are astonishing figures that prompted us to ask two questions: who is earning calculus credit in high school and what courses do students take in high school that may lead to earning calculus credit?

We pursue these questions because we suspect that there is a large proportion of students who do not earn calculus credit in high school, but who have intentions of pursuing a STEM degree. This implies that higher education institutions, and in particular mathematics departments, have to assume the responsibility of preparing the large proportion of students with much weaker mathematical preparation.

At least two reasons support our suspicions. First there are no national standards regarding the number of mathematics courses that students must complete to receive a high school diploma. Twenty-four states (including the District of Columbia) require three Carnegie units[8] of mathematics for graduation (two courses in algebra and one course in geometry); five states require four; two have differential mandates, four units for college bound students and three units for non college bound; six states have no policies; and 14 require only two units (Education Commission of the States, 2016). And second, schools do not have equal access to resources that allow them to offer equal opportunities to their students. Schools with large proportions of low-income, underrepresented minority students tend to have a lower proportion of certified mathematics teachers than schools with more affluent and Caucasian majority students (Hill and Dalton, 2013), fewer qualified counselors that would help students in their plans towards college enrolment (Engberg and Gilbert, 2014), or engage in overt and covert tracking practices that tend to disproportionately place underrepresented and low-income students in math courses that do not prepare them for STEM majors (Lee and Burkam, 2003; Tate and Rousseau, 2002).

To investigate who is earning calculus credit in college and what are the patterns of courses that may lead to such credit, we turned to the High School Longitudinal Study (HSLS). The HSLS represents an unprecedented effort to gather longitudinal data from students starting in 9th grade that can shed light into course-taking patterns and performance and that can help uncover the pathways leading to calculus and beyond. It also provides a rich collection of data about students, schools, and instruction that can ultimately help us to characterize students at the end of their high school years. Different from other longitudinal studies, HSLS includes several important innovations: systematically recorded high school transcripts; standardized measures of students' knowledge of mathematics; parent, teacher, counselor, and administrator surveys, which allow better understanding of students' family, school, and communities; and student survey data reflecting their self-efficacy, interest, and motivation in mathematics and other domains. As such, this data set can be useful to understand the paths students take through secondary and postsecondary education, especially how those paths lead toward or away from calculus.

The paper is organized into four sections. We present a brief review of literature that helps identify some variables that can play a role in who earns calculus credit. We then describe the data set, the variables chosen, and the analysis performed. After presenting the findings we conclude with suggestions for further analyses.

Literature Review

Mathematics course taking is an important predictor of student achievement in schools and beyond (Bryk, Lee, and Smith, 1990; Marion and Coladarci, 1993; Sadler and Tai, 2001; Tyson, Lee, Borman, and Hanson, 2007). We found several features continually associated with mathematics course taking: gender, ethnicity, motivation, and school organization.

Gender and ethnicity are the most common variables included in these analyses, driven by the interest in increasing the number of students from underrepresented groups in STEM fields (women, African American, Hispanic/Latino, Native American) to pursue those degrees. A number of studies indicate that female students are more likely than males to take fewer mathematics courses in high school and less likely than males to take advanced mathematics courses, in spite of females performing better in these courses

8 One unit reflects one year of coursework. See: ecs.force.com/mbdata/mbprofall?Rep=HS01

(Benbow and Stanley, 1982; Davenport et al., 1998; Maple and Stage, 1991; Updegraff, Eccles, Barber, and O'brien, 1996). Similarly, numerous studies that account for students' ethnicity document similar patterns for African American/Black and Hispanic/Latino/a students, even when aspects such as socioeconomic status (SES) or prior academic achievement has been controlled for (Davenport et al., 1998; Riegle-Crumb, 2006; Riegle-Crumb and Grodsky, 2010; Tyson et al., 2007). The wide achievement gap between Caucasian and Asian students and students from other ethnic backgrounds starts in middle school and tends to widen as students progress through high school (Lubienski, 2002; Moses and Cobb, 2001; Riegle-Crumb, 2006; Riegle-Crumb and Grodsky, 2010).

Students' motivation, in particular their self-efficacy beliefs (one's self-perceived ability to successfully achieve specific goals in defined contexts) has also been closely tied to students' mathematics course taking patterns in high school (Updegraff et al., 1996) and is significant even after controlling for prior achievement and background differences (Chen and Zimmerman, 2007). While one can't influence students' gender or ethnicity, students' mathematics self-efficacy is a personal attribute that is malleable; that is, it can change in response to specific interventions that seek to teach students about how they approach their work (Dweck, 1986, 2006) .

Beyond student attributes, school resources and their organization have also been strongly associated with course taking patterns, which in turn are associated with drop out and graduation rates, and four-year and two-year college enrolment (Bryk et al., 1990; Croninger and Lee, 2001; Engberg and Gilbert, 2014; Grubb, 2008; Lee and Burkam, 2003; Lee, Croninger, and Smith, 1997).

Using this brief literature as a base, but with a particular focus on factors most likely to be associated with achievement in high school calculus, we formulated three specific research questions:

1. What mathematics classes do U.S. students complete in high school?
2. What student characteristics (e.g., sex, race, SES, 9th grade mathematics score, 9th grade mathematics course, and mathematics self-efficacy) relate to students' completion of mathematics courses?
3. To what extent can student characteristics in 9th grade be used to predict the likelihood of completing calculus in high school?

Together these questions help us understand who is earning calculus credit in high school and which courses students taken in high school may lead to earning calculus credit. For this investigation we did not pursue aspects of school organization that may also contribute to course taking patterns, as our main goal was to build a foundation for further mining of the data set. Additionally, aspects of instruction, which necessarily are tied to what students experience in their classroom, were not included in this work. As we state in our final section, the next steps in our work will consider these and other potentially useful variables that can provide a better picture and understanding of the mathematical courses students have once they finish high school.

Methods

Data Source

The first three waves from the High School Longitudinal Study of 2009 (HSLS:09), as published by the National Center for Education Statistics (NCES, 2015), were used to explore the research questions. The HSLS:09 is a nationally representative cohort study of 9th graders in public and private high schools in the United States designed to follow students through high school and into postsecondary education and the workforce. The first wave of data collection took place during the fall of the 2009-2010 school year, when students were in the 9th grade (their first year of high school). A follow-up was completed in 2011 when most students were in the spring term of 11th grade (their third year of high school), and an update was completed in 2013 (their first year after high school). A second follow-up is planned for 2016 (three years after the expected graduation year) to learn about students' postsecondary experiences, and again in 2021 to learn about participants' choices, decisions, attainment, and experiences in adulthood. In addition, the study includes high school transcript data collected in 2013–2014 that provides systematic information on all the mathematics and science courses the students took. The study includes interest and motivation items in the student questionnaire, with the anticipation that such data can help provide more accurate measures of key factors predicting students' choice of postsecondary paths, including majors and eventual careers.

Sample

The sample design was a stratified, two-stage, random sample with schools selected at the first stage and students within those schools selected at the second stage. Hence, the sample is nationally representative of 9th graders in 2009–2010 and of schools with 9th and 11th graders in 2009. 23,503 ninth graders in 940 schools completed the base year HSLS:09 survey. The multi-stage design frame allows for accurate statistical generalization to the more than 4.2 million students attending over 23,000 high schools in the United States during the study period. The study includes a math assessment and survey component in the fall of 9th grade (2009) and again in the spring of most students' 11th grade year (2012). Students who do not complete high school were followed with certainty and surveyed again at the same time as the rest of the sample who remained in school.

Altogether there are ten data protocols included in the sample (Base year: Student, Parent, Teacher, School Counselor, and School Administrator; First Follow-Up: Student, Parent, School Questionnaire, School Administrator; 2013 Update: Student, Parent). In all, they contain 5,818 variables with publicly available data, and approximately 800 variables with blinded or omitted data (location, school name, ethnicity). We focused our analysis on students for whom data is available in each of the first three waves of data collection plus the transcript study, which includes $N = 15{,}188$ individual student records.

Procedures

We began the investigation by selecting 111 variables from the dataset with particular relevance to our research questions, organized into constructs (e.g., indicators of SES, self-efficacy beliefs, prior achievement, course completions, demographic characteristics). This restricted dataset contains more than 1,446,285 non-missing values (86% complete). We summarized the variables using basic univariate and bivariate descriptive statistics, using statistical summaries, cross-tabulations, boxplots, histograms, scatterplots, and correlation tables to help to identify the center, spread, and shape of the distributions, as well as variables associated with students' completion of high school calculus.

Framing students' sequences of completed courses as the dependent variable in our analysis,[9] we visualized the marginal distributions using proportional flow diagrams (also called Sankey diagrams, see Riehmann, Hanfler, and Froehlich, 2005), which allow for visual representation of differences among temporally ordered factors for subsets of the data. In cases in which continuous variables were associated with course completion patterns, we transformed the variables to ordinal levels by quartile, so that scores in the top quartile were labeled as "high," scores in the middle quartiles labeled "medium," and scores in the bottom quartile labeled "low."

In the final stage of the analysis, we used standard logistic regression modeling procedures, with the probability of completing high school calculus as the response variable, and those variables most closely associated with differences in high school course patterns as potential explanatory variables. Models were fitted and validated using a "train-test" modeling strategy in which the model was fitted to the data using a two-thirds random sample, reserving the final one-third of data as an external test for the classification accuracy of the logistic model. Specifically, we implemented an algorithm in which (1) two-thirds of the data is selected at random (i.e., the "training data"), (2) the predefined model is fitted to the training data, (3) a probability threshold is selected to balance false classifications on the training data, (4) the fitted model is applied to

9 For the purposes of this report we say that a student has "completed" a high school mathematics course if and only if the student earned credit for the course according to their high school transcript. The students' individual grades in the courses, along with the specific content of the course and grading criteria are not available in our data set. Notably, this means we have no indications of whether a calculus course completed by a student may or may not have been completed as part of a dual-enrollment, concurrent enrollment, AP, or IB calculus program.

the other third of the data (i.e., the "testing data") using the probability threshold for binary classification, and (5) the predicted classifications for the testing data are compared to the actual course completions of the students. The accuracy of the classifications were aggregated over 1,000 simulations of the train-test algorithm, yielding information about the predictive validity of the logistic regression model for the given 50% classification threshold. A total of eight models (incorporating differing assumed interaction effects) were considered in the analysis, with the best performing model presented in the results. The only variable considered in the modeling that is not included in the final model, was students' sex, due to a weak observed association with the response variable and a very small (and statistically insignificant) observed main effect size in the logistic regression model.

Measures

Most of the measures included in our analysis are typical in the education literature (readers in search of more detailed descriptions of the measures are encouraged to consult the HSLS project) but we note two important caveats. First, there are multiple variables available in the HSLS data set that are derived from students' performance on a 9th grade mathematics exam. The test, which covers six domains of algebraic content and four algebraic processes, was administered by computer using a two-stage design and scored through item response procedures. Though several types of scores are generated from this process (in the language of IRT, these include raw ability scores, norm-referenced ability scores, estimated number correct scores, etc.), we used the normalized "ability scores," which represent a norm-referenced estimate of students' mathematics knowledge in relation to their peers and are scaled to an approximately normal distribution.

Second, the sequence in which students complete high school mathematics courses can vary greatly, and the HSLS data set does not include temporal information about when the students in the sample completed the respective mathematics courses. Nonetheless, there is a predictable order in which students may complete high school mathematics curricula (e.g., Algebra I almost always comes before Algebra II). Consequently, we assumed that students would have completed high school mathematics courses in the following order: Algebra I (ALG1), Geometry (GEO), Algebra II (ALG2), Integrated Mathematics (INTEG), Precalculus (PRECALC), Trigonometry (TRIG), Statistics (STAT), Calculus (CALC). While this ordering appears to be valid in the aggregate based on our analysis, there is much variation across states, school districts, and even schools and individuals, so any ordering of mathematics courses appearing in the results should be interpreted with this caveat in mind.

Results

Research Question 1: *What mathematics classes do U.S. students complete in high school?*

Based on the sample (N = 15,188) of U.S. students' high school transcripts, nearly all students earned credit for Algebra I (96%), the vast majority completed Geometry (78%), and the majority completed Algebra II (62%). Further, many completed Precalculus (34%), Trigonometry (16%), Statistics (11%), and/or an Integrated Mathematics course (7%). Approximately one in five students earned high school credit for Calculus (19%). Figure 1 illustrates the overall sequences of mathematics courses completed by students in the sample. The ribbons connecting the labeled courses are proportional to the number of observed students who completed the given sequences of courses. Courses labeled numerically (e.g., "C3") represent the place of the course within the respective students' ordered list of completed mathematics courses. To facilitate the reading of the flow diagrams, we include a table that has the percentage represented in the diagram, as needed.

Research Question 2: *What student characteristics (sex, race, SES, 9th grade math score, 9th grade math course, and self-efficacy) relate to students' completion of mathematics courses?*

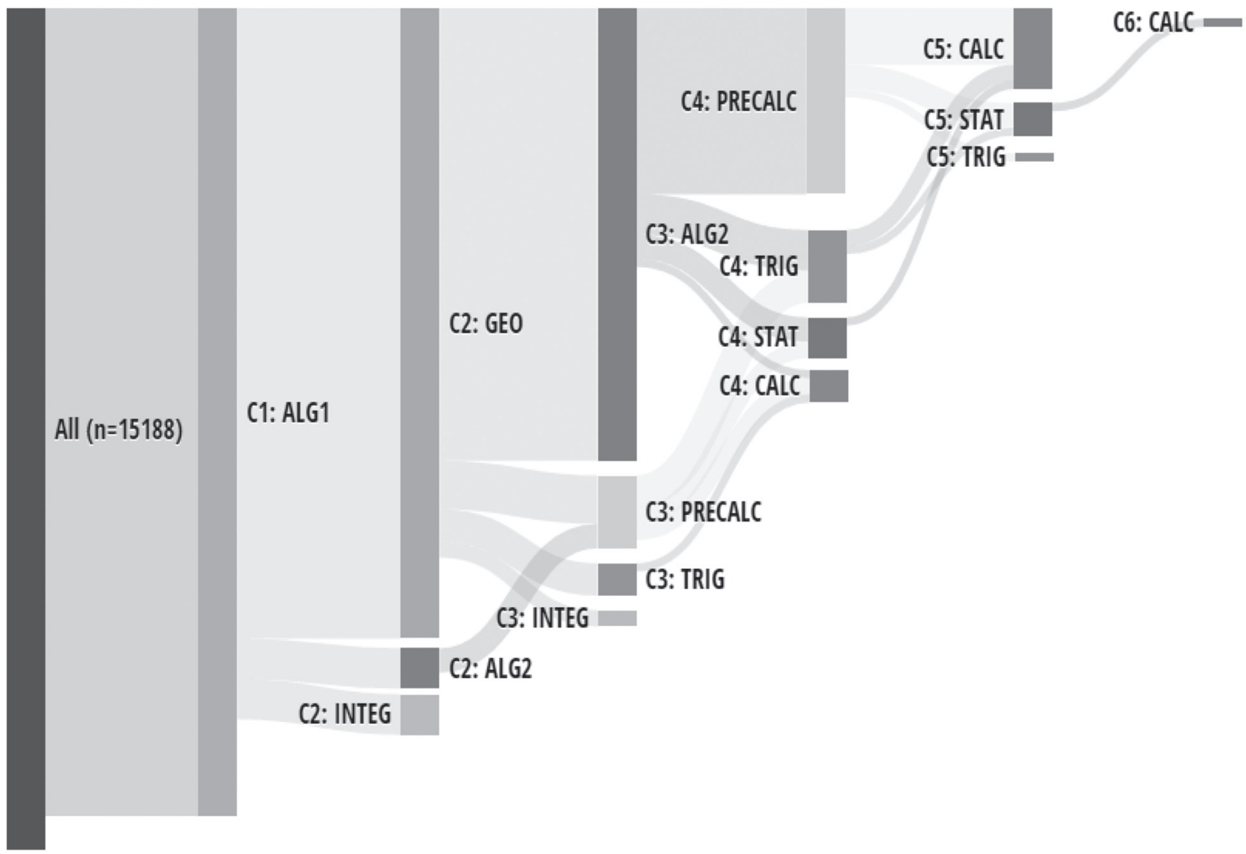


Figure 1. Proportional flow diagram of U.S. high school mathematics courses.

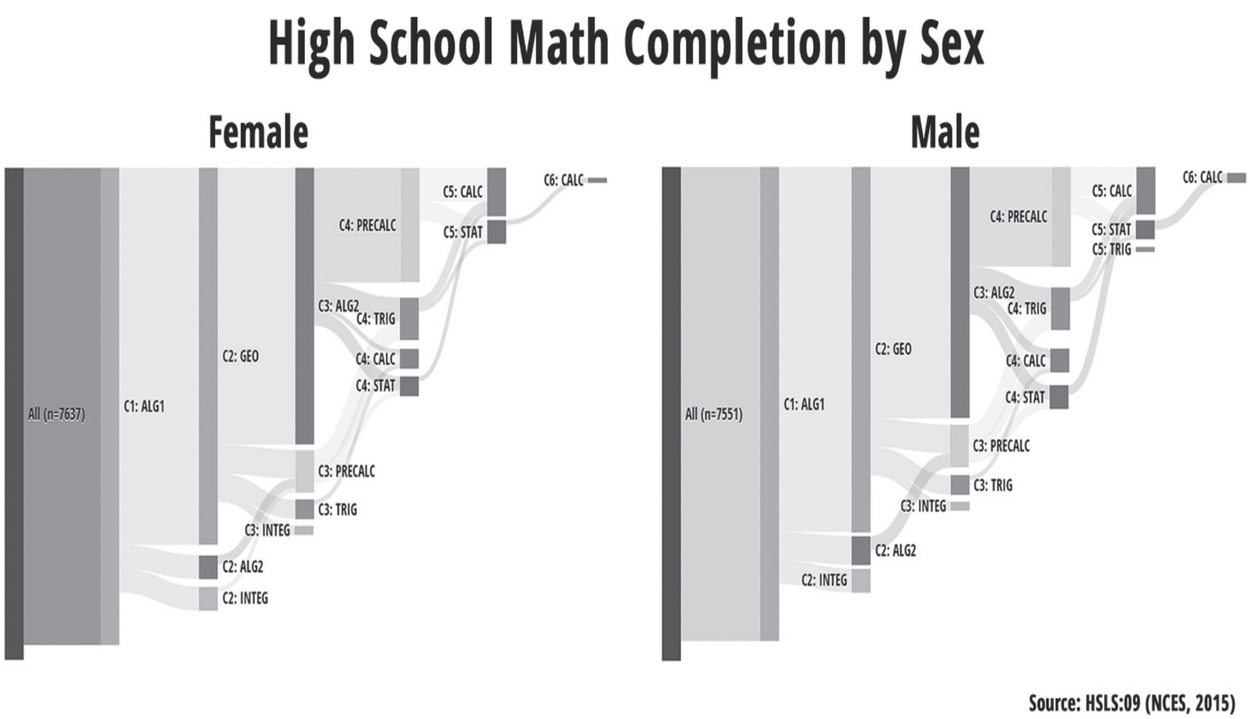


Figure 2. Proportional flow diagrams of high school mathematics courses by students' sex.

Sex

Course taking patterns were similar among male and female students in the sample. As Table 1 indicates, female students were as likely or more likely as their male counterparts to complete all types of mathematics courses, with the exception of Statistics and Calculus.

Course completions among male and female students are illustrated in Figure 2.

Table 1. High school mathematics course completion by gender (N = 15,188)

	N	Alg1	Geo	Integ	Alg2	Precalc	Trig	Stat	Calc
Male	7,551	96%	77%	7%	60%	32%	15%	12%	19%
Female	7,637	97%	79%	7%	64%	35%	16%	11%	18%

Race

High school mathematics course completion rates varied substantially by students' self-reported race. In particular, Asian students were much more likely than non-Asian students to complete Precalculus, Statistics, and Calculus. Black/African American students were the least likely to complete Calculus during high school (just 8%; see Table 2).

Table 2. Proportion of students completing mathematics courses by race (N = 15,188)

	N (%)	Alg1	Geo	Integ	Alg2	Precalc	Trig	Stat	Calc
Asian	1,229 (8%)	99%	76%	6%	67%	55%	17%	22%	47%
Black	1,511 (10%)	94%	74%	10%	55%	22%	14%	8%	8%
Hispanic/ Latino	2,307 (15%)	95%	79%	6%	58%	25%	12%	8%	12%
White	8,649 (57%)	97%	79%	7%	65%	36%	17%	12%	19%
More than one	1,326 (9%)	96%	76%	7%	60%	30%	14%	9%	16%
Other	166 (1%)	90%	75%	4%	54%	19%	14%	8%	13%

The observed differences in overall course completion are evident in Figure 3, which shows a relatively larger proportion of Asian students completing many more mathematics courses than their peers in other racial groups.

Socioeconomic Status (SES)

There were large observed differences in course taking patterns for students of different SES. In particular, there is a strong association between SES and completion of courses beyond Algebra I, with students of low SES appearing to be less likely to complete Precalculus, Statistics, and Calculus (see Table 3). The apparent effects of SES on course completions is displayed in Figure 4 which shows the increasing complexity of course patterns and relative increases in the numbers of students completing advanced mathematics classes among students of higher SES.

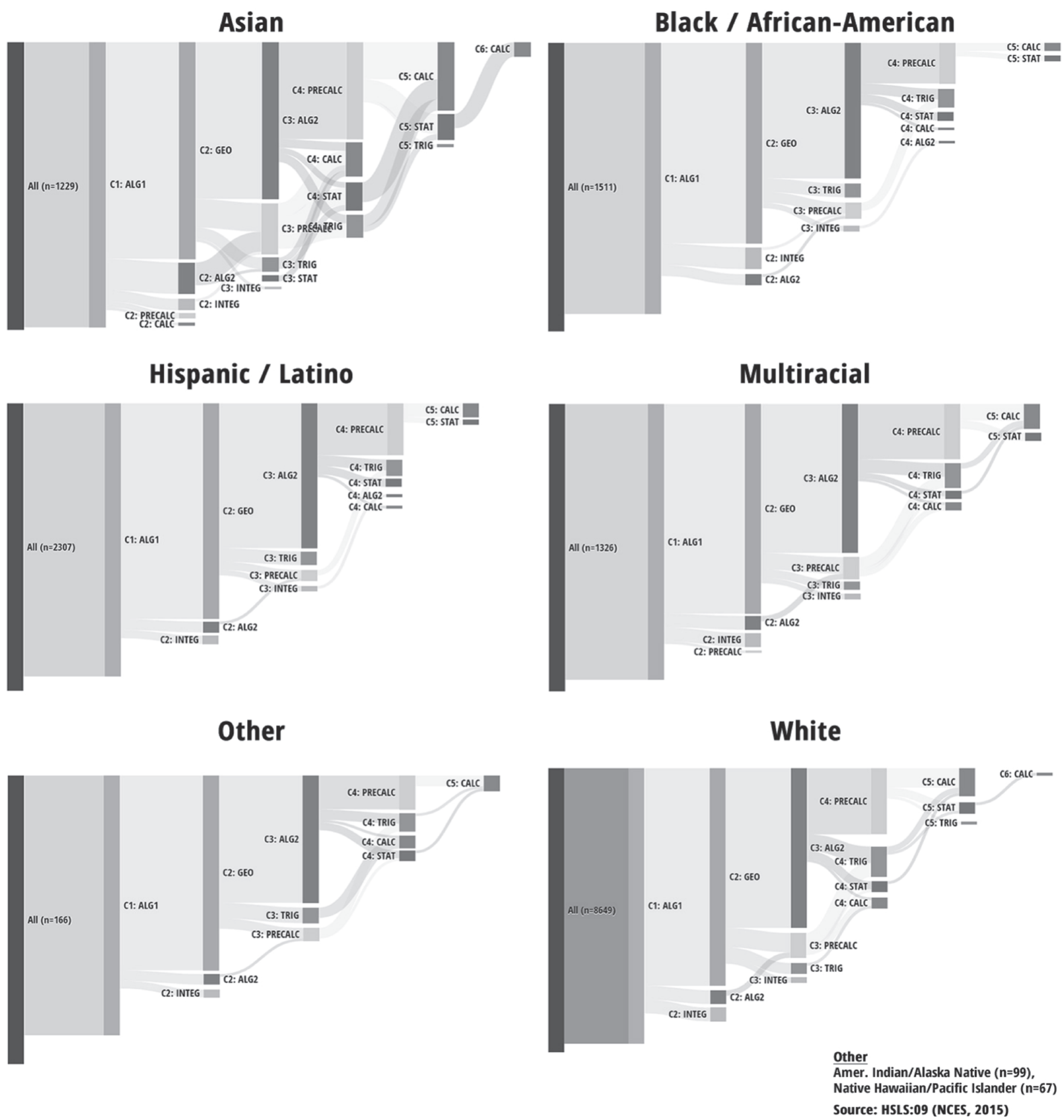


Figure 3. Proportional flow diagrams of completion of high school mathematics courses by race.

Table 3. Proportion of students completing the course sequences by SES. (N = 15,187)

	N (%)	Alg1	Geo	Integ	Alg2	Precalc	Trig	Stat	Calc
Low	3,787	92%	71%	8%	52%	18%	10%	6%	7%
Medium	7,601	97%	81%	7%	64%	31%	16%	11%	15%
High	3,799	99%	81%	6%	70%	55%	21%	19%	38%

Note: Low (1st quartile) < –0.45; Medium (2nd and 3rd quartiles): –0.45 < SES < 0.66; High (4th quartile): SES > 0.66.

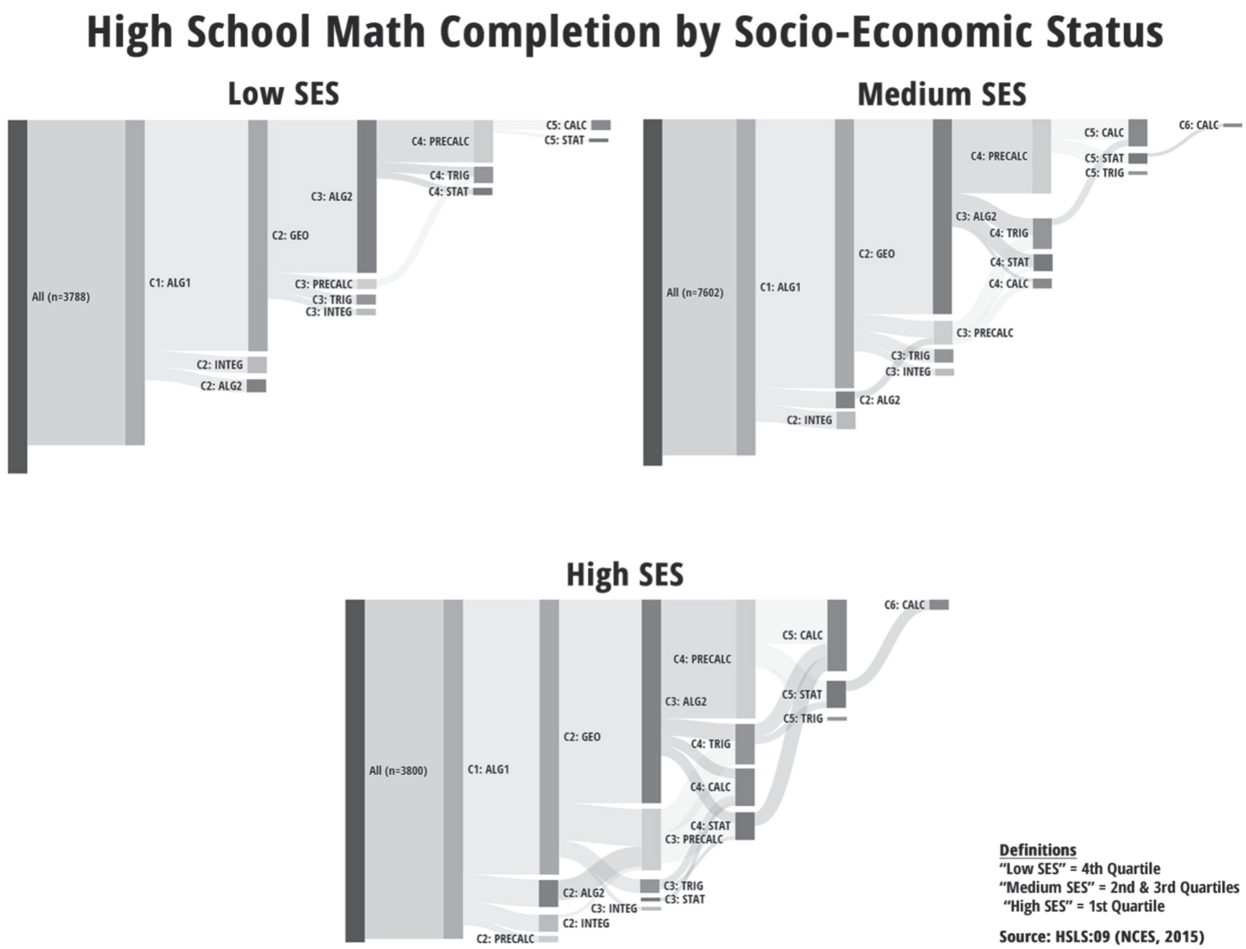


Figure 4. Proportional flow diagrams of high school mathematics courses by students' SES.

Knowledge of Mathematics in 9th grade

The direct measure of students' mathematics knowledge in Grade 9 was strongly associated with their completion of mathematics courses of all types in high school, with the exception of Integrated Mathematics. For instance, while just 2% of students in the bottom quartile of mathematics performance in Grade 9 completed high school calculus, 50% of those in the top quartile completed high school calculus. Notice also that only 48% of students in the low quartile complete Algebra 2 in high school, and that less than 10% complete Precalculus or Trigonometry. See Figure 5 and Table 4.

Table 4. Proportion of course completion by level of 9th grade math score (N = 15,188)

	N	Alg1	Geo	Integ	Alg2	Precalc	Trig	Stat	Calc
Low	3,797 (25%)	90%	68%	7%	48%	9%	8%	4%	2%
Medium	7,594 (50%)	98%	83%	7%	67%	31%	17%	10%	11%
High	3,797 (25%)	100%	78%	5%	69%	63%	22%	21%	50%

Note: Low: 9th grade math score < –0.43; Medium: –0.43 < 9th grade math score < 0.81; High: 9th grade math score > 0.81.

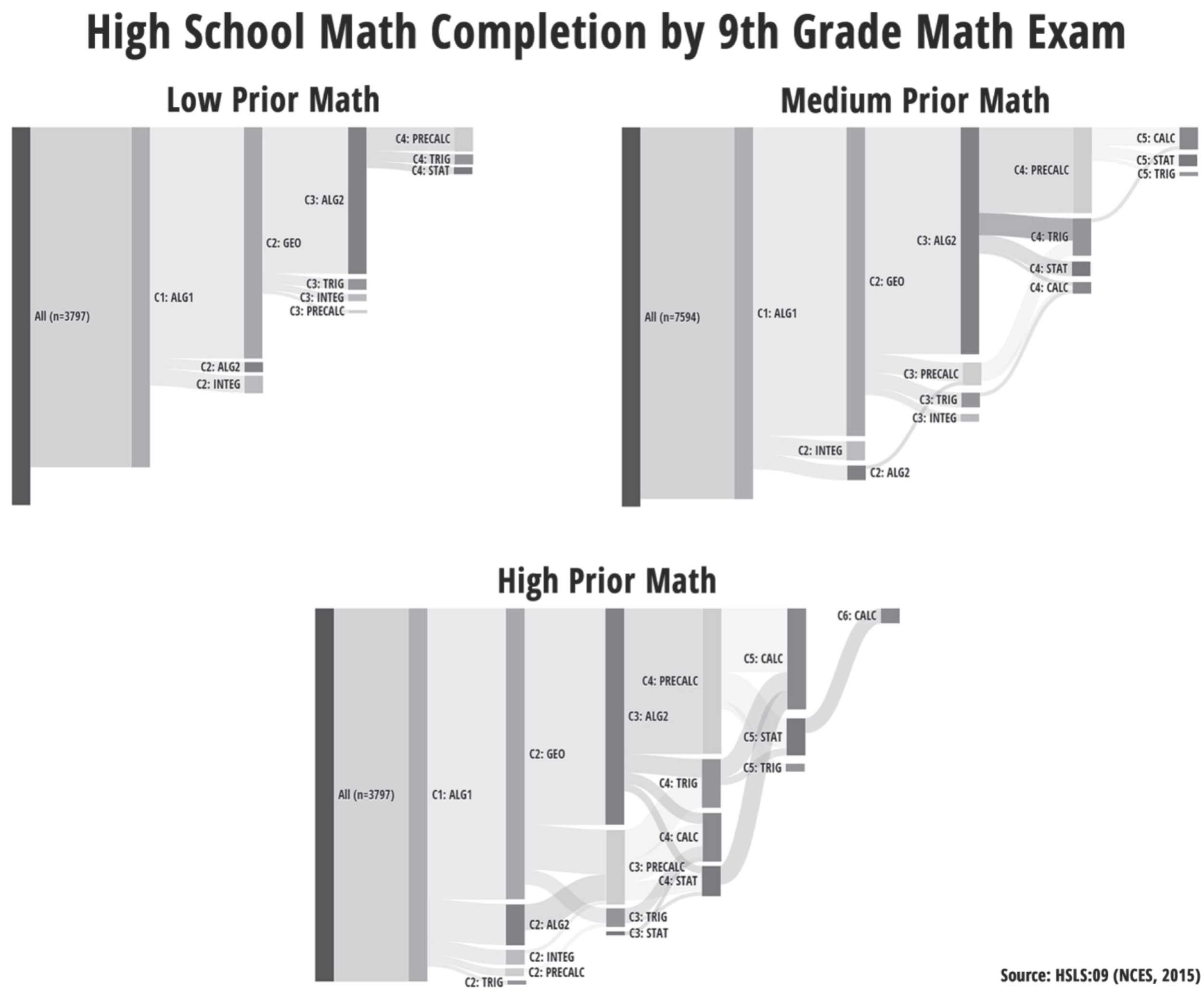


Figure 5. Proportional flow diagrams of high school mathematics courses by students' 9th grade mathematics knowledge.

Grade 9 Mathematics Course

Students' mathematics course in Grade 9 was strongly associated with their subsequent mathematics course completion. More than half (54%) of students enrolled in a mathematics course above Algebra I in Grade 9 completed Precalculus, compared to just 24% of those enrolled in Algebra I (see Table 5). As indicated in Figure 6, even among students who did complete calculus, students' placement in Grade 9 was linked to the courses they completed in addition to calculus, with a small proportion of students placed above Algebra I completing Geometry, and a higher proportion completing Precalculus.

Table 5. Proportion of students completing math courses by level of course taken in 9th grade. (N = 14,740)

	N	Alg1	Geo	Integ	Alg2	Precalc	Trig	Stat	Calc
Below	1,788 (12%)	76%	53%	5%	32%	5%	5%	3%	2%
Algebra 1	7,249 (50%)	99%	87%	2%	67%	24%	16%	7%	5%
Above	5,703 (39%)	100%	76%	14%	67%	54%	19%	20%	41%

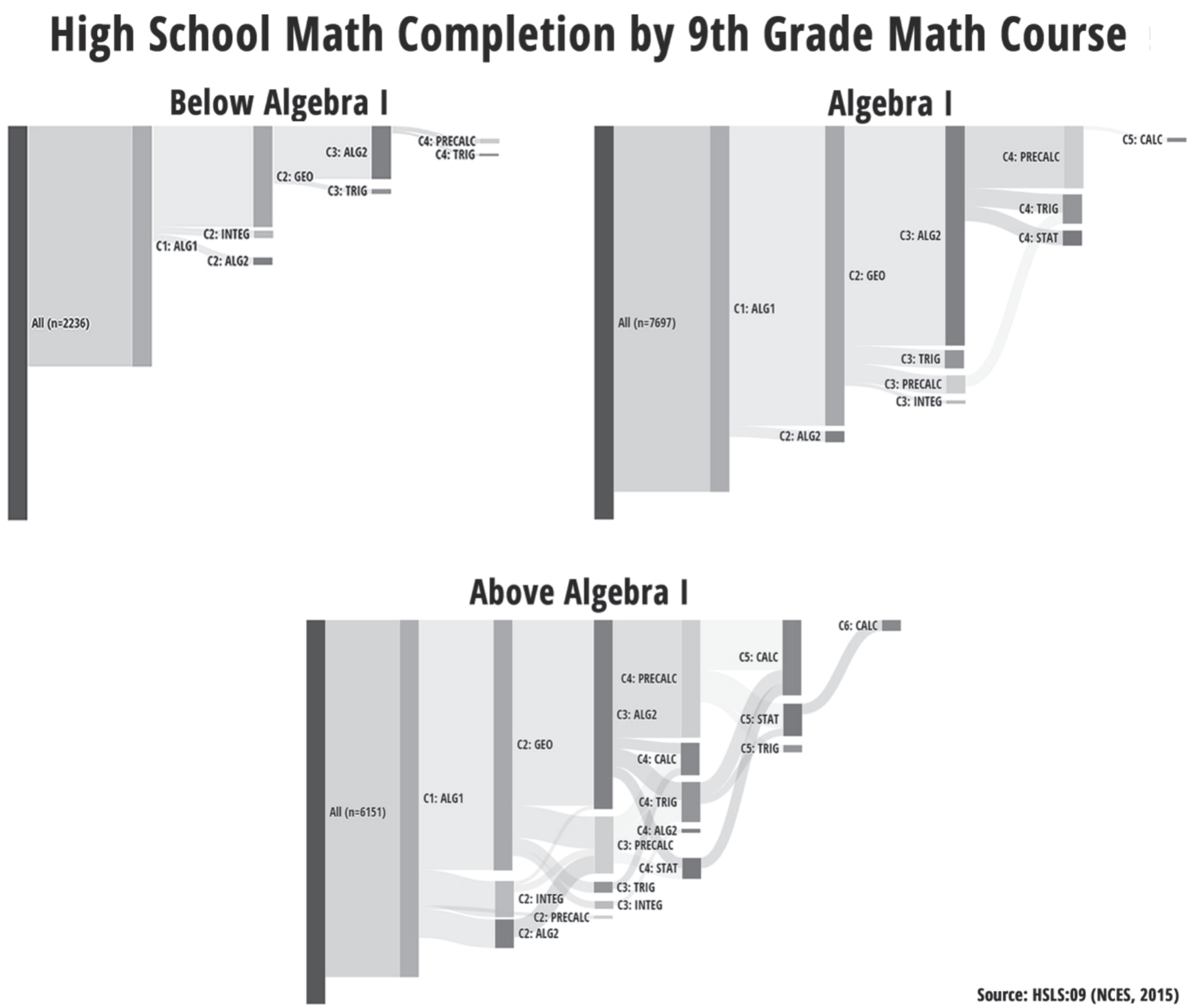


Figure 6. Proportional flow diagrams of high school mathematics courses by students' 9th grade mathematics course.

Mathematics Self-Efficacy

Students with higher reported self-efficacy in mathematics in Grade 9 were more likely to complete nearly all types of mathematics courses than those with lower reported mathematics self-efficacy (see Table 6). The proportion of students in the top quartile of self-efficacy in Grade 9 who completed calculus in high school (32%) is more than three times the proportion of students who were in the bottom quartile of self-efficacy in Grade 9 (9%) and almost double of those students who were in the middle of the distribution (Figure 7).

Table 6. Proportions of courses completed by level of self-efficacy (N = 13,368)

	Alg1	Geo	Integ	Alg2	Precalc	Trig	Stat	Calc	*N* (%)
Low	94%	74%	7%	57%	22%	13%	10%	9%	3,306 (25%)
Medium	97%	80%	7%	64%	36%	16%	11%	19%	6,711 (50%)
High	99%	82%	5%	67%	46%	20%	14%	32%	3,351 (25%)

Note: Low (1st quartile): self-efficacy < –0.34; Medium (2nd and 3rd quartile): –0.34 < self-efficacy < 0.78; High (4th quartile): self-efficacy > 0.78.

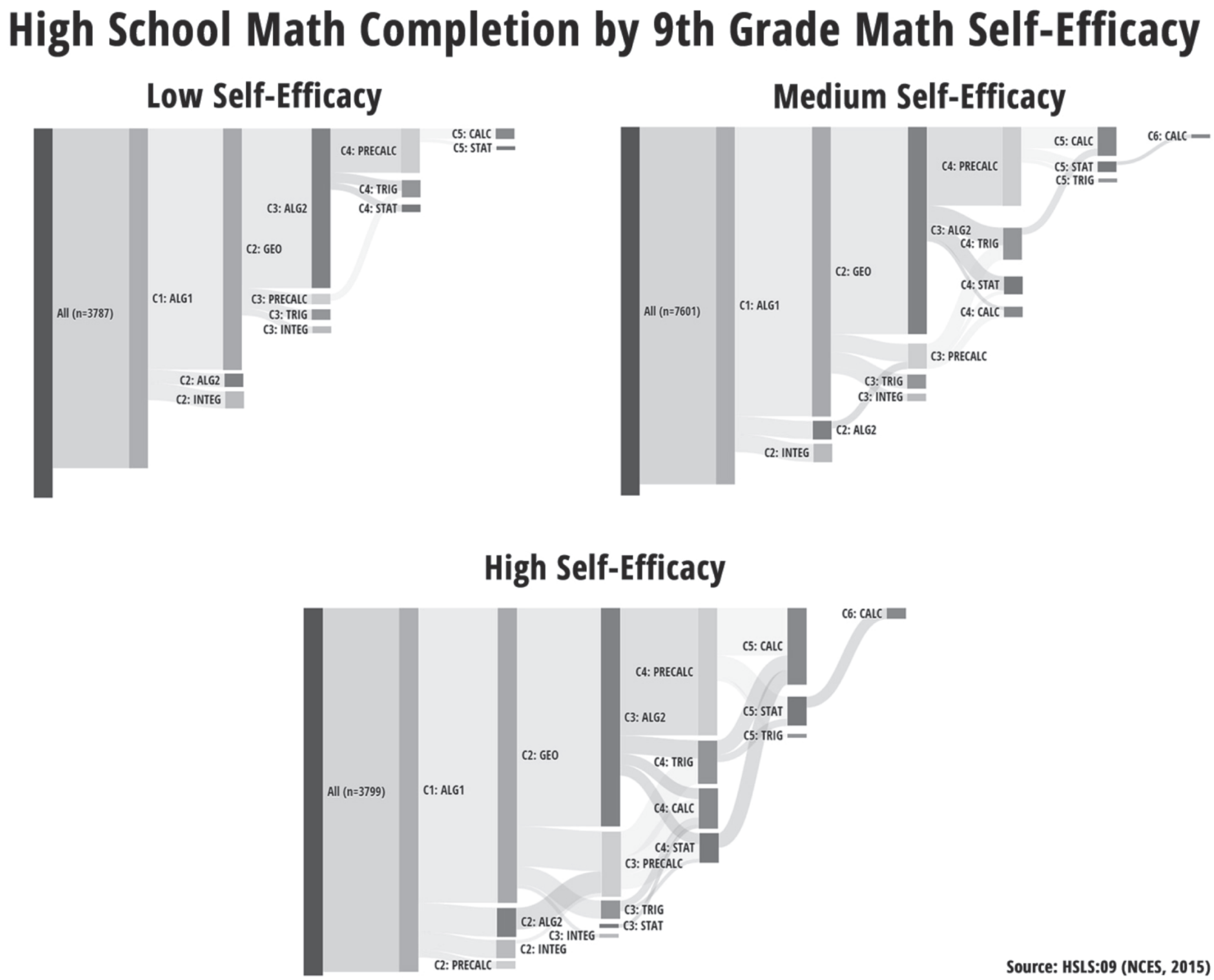


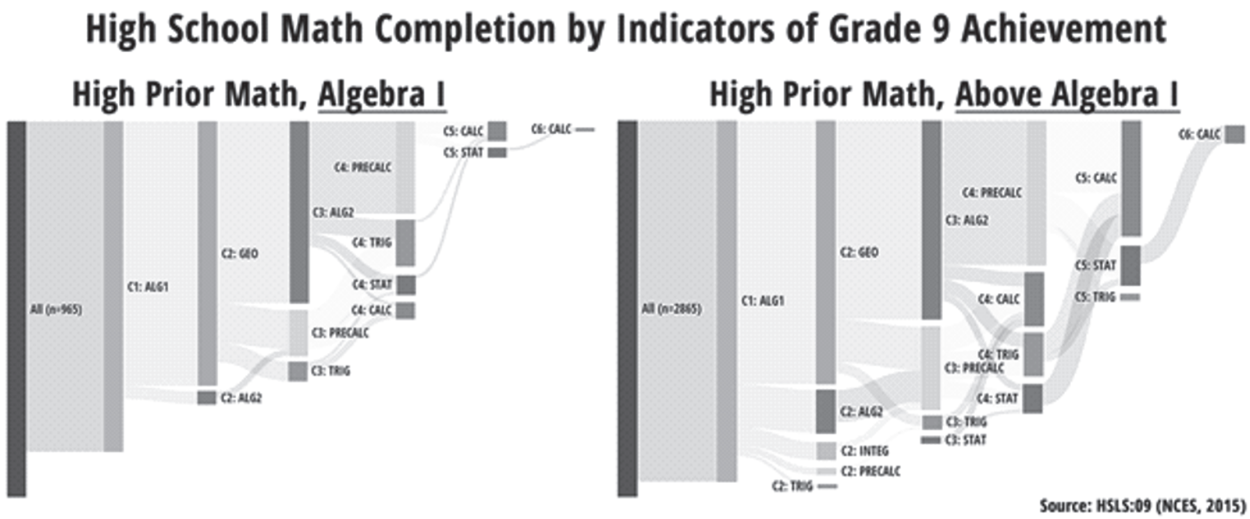
Figure 7. Proportional flow diagrams of high school mathematics courses by students' 9th grade mathematics self-efficacy.

Interactions among Student Characteristics

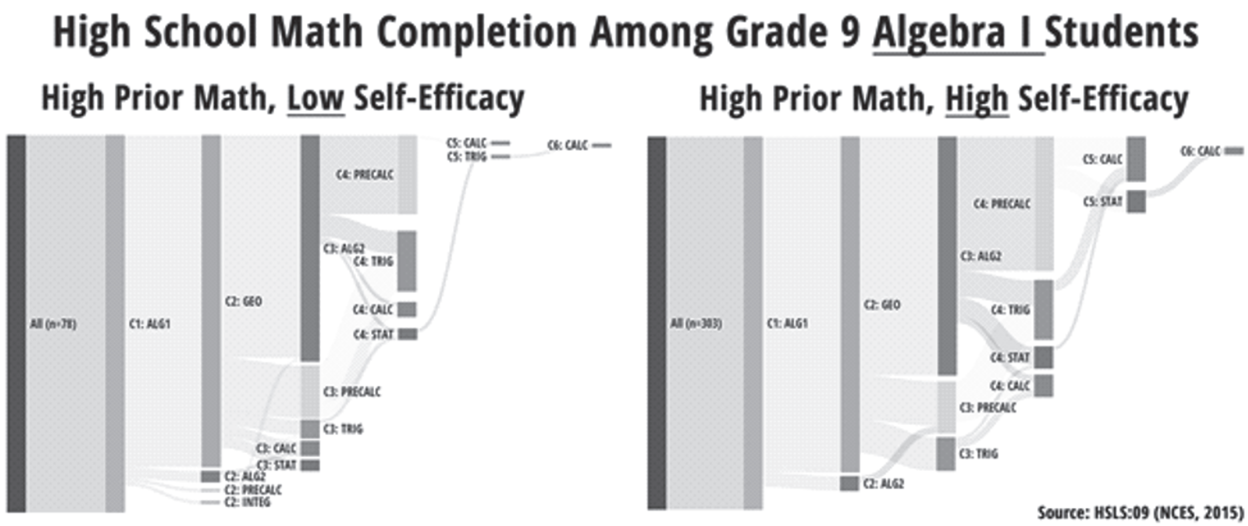
Because high school mathematics exists within a complex milieu of personal, social, and cultural factors, it is important to consider potential indications of interaction effects. The large and representative data sample allows for consideration of two- and three-way interactions among the factors with counts of students in each sub- or sub-sub-category exceeding 100 in most cases. Figure 8 shows examples of apparent interactions among the student characteristic variables. Specifically,

- Among students scoring in the top quartile for the measure of mathematics knowledge in Grade 9, those placed in Algebra I in Grade 9 were less likely to complete more than three mathematics courses, particularly a sequence that included Calculus (Figure 8a)
- Among the students placed in Algebra I in Grade 9 who scored in the top quartile on the measure of mathematics knowledge in Grade 9, those with high self-reported mathematics self-efficacy in Grade 9 completed more courses, including Calculus, than comparable peers who reported low mathematics self-efficacy in Grade 9 (Figure 8b).

Research Question 3: *To what extent can student characteristics in 9th grade be used to predict the likelihood of completing calculus in high school?*



(a)



(b)

Figure 8. Patterns of course taking by (a) high scoring Grade 9 mathematics students with different Grade 9 mathematics placement and (b) high scoring Grade 9 Algebra I students with different mathematics self-efficacy.

In order to address the research question regarding the extent to which Grade 9 data can be used to effectively predict students' completion of high school calculus, we fit a logistic regression model with the log-odds of completing calculus as the response variable, and the relevant student characteristics from the prior research question as the explanatory variables. The final fitted model was

$$\log\left(\frac{P(CALC_i)}{1-P(CALC_i)}\right) = \mathrm{PRIORMATH}_i + \mathrm{PLACEMENT}_i + \mathrm{PRIORMATH}_i \times \mathrm{PLACEMENT}_i + \mathrm{SES}_i + \mathrm{SELFEFF}_i + \mathrm{RACE}_i + \beta + \text{error},$$

where CALC is the binary outcome of completing calculus (or not), PRIORMATH is the student's composite score on the Grade 9 mathematics score, PLACEMENT is the three-level ordinal variable for the students' Grade 9 mathematics course (Below Algebra I, Algebra I, Above Algebra I), SES is the composite measure of socioeconomic status, SELFEFF is the composite self-reported Grade 9 mathematics self-efficacy, RACE is the binary variable indicating whether the student self-reported their race as Asian, and β is the constant (intercept).

The estimated main and interaction effects in the logistic regression model are presented in Table 7 as odds ratios. The intercept gives a baseline estimate for the odds of completing calculus (~2%), with the listed estimates serving as multiplicative factors (between 0 and 1 means reduced likelihood, greater than 1 means increased). The following examples illustrate how to interpret the estimated odds ratios:

- holding the other variables at the baseline (Algebra 1 placement, non-Asian, medium SES, medium self-efficacy), a student with high prior math knowledge (1 standard deviation above the mean on the 9th grade exam) has approximately .08:1 odds (8% chance, $4.1 \times .02$) of completing calculus in high school;
- holding the other variables at the baseline, a student of high SES placed below Algebra I in 9th grade mathematics has approximately .01:1 odds (1% chance, $.02 \times 1.6$ x 0.39) of completing calculus in high school;
- holding the other variables at the baseline, an Asian student with high prior math and high SES placed Above Algebra I in 9th grade has approximately 2.87:1 odds (74% chance, $.02 \times 2.61 \times 4.15 \times 1.6 \times 8.27$) of completing calculus in high school.

Table 7. Estimated main and interaction effects in the logistic regression model.

	Estimated Odds Ratio	2.5% Est.	97.5% Est.
(Intercept)	0.02	0.02	0.03
PRIORMATH	4.15	3.42	5.07
Above Alg1	8.27	6.81	10.11
Below Alg1	0.39	0.17	0.75
SES	1.60	1.48	1.72
SELFEFF	1.43	1.34	1.52
RACE: Asian	2.61	2.19	3.11
PRIORMATH: Above Alg1	0.70	0.56	0.87
PRIORMATH: Below Alg1	2.71	1.38	5.91

The model performed well on our analysis of diagnostic accuracy. The modeling procedure identified 40% as the probability threshold associated with approximately balanced classification errors, and the "train-test" procedure resulted in an estimated overall diagnostic accuracy (predicted calculus result equaled actual result) of 86%, with a 7% false negative rate (predict = did not complete calculus, actual = completed calculus) and a 8% false positive rate (predict = complete calculus, actual = did not complete calculus). For reference, the best single-variable model had a diagnostic accuracy of 64%, with a 3% false negative rate and a 34% false positive.

Discussion and Suggestions for Further Work

Our analysis suggests that U.S. students' race, socio-economic status, prior mathematics knowledge, mathematics course placement, and mathematics self-efficacy each has an important role in the likelihood of completing calculus in high school. Though the analysis is strictly observational (HSLS tests no interventions), the large and representative sample suggests changes in any combination of the factors is likely to affect high school enrollment in and completion of calculus. For example, those interested in increasing high school calculus enrollment may pursue policies and programs that lead to increased Algebra I completion prior to 9th grade and increased mathematics self-efficacy. Other strategies may include providing support for underrepresented minority students and students with low SES to stay on track for courses that lead to completing calculus or creating alternative pathways to the standard high school mathematics course sequence (e.g., Algebra I, Geometry, Precalculus instead of Algebra I, Geometry, Algebra II, Precalculus). This latter strategy needs to be closely connected to the varying (often statutory) high school mathematics requirements across states.

Our results are also useful for institutions of higher education, at which calculus has been a traditional mainstay of entry-level mathematics. The findings suggest that approximately 19% of high school students have completed calculus, with large variations across different student populations (especially for Asian students and for students who have high completed Algebra 1 by grade 8). The results also suggest that those students pursuing calculus in college who have not completed calculus in high school have a different statistical profile: first-time calculus takers in college are more likely to belong to historically underrepresented groups, less likely to have completed more than three high school mathematics courses, and more likely to have weaker knowledge of mathematics by their 9th grade (suggesting a relatively weaker middle school mathematics preparation). That is, statistically speaking, typical students taking calculus for the first time in college appear to be much more likely to have had a history of lower achievement in mathematics. They may benefit most from academic and social support.

Both the proportional flow diagrams and the statistical modeling results suggest that 9th grade mathematics (course taken, mathematical knowledge) is very strongly associated with high school calculus completion which might be seen as a serious concern. For example, it is very unlikely for students to complete high school calculus if they have not completed Algebra I prior to 9th grade. When we consider that there are also large differences associated with race and SES, the findings support, with very few exceptions, the notion that there is a broad de facto system of tracking in U.S. high schools that may contribute to widening achievement gaps in mathematics and differential access to STEM fields.

To our knowledge, this report contains the first attempts to conduct large-scale statistical modeling of calculus completion among U.S. high school students. The analysis was made possible by publicly released interim data from HSLS, which contains many variables that may potentially add to and refine our analysis (e.g., school location). In future analysis, we plan to extend the analysis of HSLS to a conceptually driven, detailed investigation of school and instructional factors. We also plan to request access to data about calculus completion not available in the public data set, including students' participation in IB, AP, and related programs, and students' geographic location. A conceptually driven approach, informed by the literature and by feedback from experts on high school calculus, will allow us to pursue finer grain hypotheses regarding the role that school resources play in assisting students in their decisions to pursue calculus, as well as potential effects of instructional characteristics in shaping students' high school mathematics achievement. We are moderately optimistic that instruction in particular might be found to have a strong influence on students' mathematics achievement.

The next round of data from the HSLS data, planned for spring 2016, will also provide important data by looking at how high school students transition into postsecondary education. The upcoming data will provide large-scale longitudinal evidence to answer questions like "How is postsecondary education different for students who do and do not complete calculus in high school?" and especially "Does completing high school calculus affect students' choices and success in pursuing a STEM field?"

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Putting Brakes on the Rush to AP[10] Calculus

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This article is an abbreviated version of two unpublished articles[11] that reported on two separate but related studies that involved different cohorts of Rutgers students. "The Rush to Calculus" by the first author involved an analysis of students' high school and college transcripts to determine how many students achieve advanced placement in calculus and how many maintain their advanced placement in their first year of college (hint: not too many do). The second study, "Why Do Students Rush to Calculus?" by both authors surveyed seniors who had reported their AP exam scores to the university in order to investigate why they took AP Calculus, what were their expectations from the course, and what were the consequences of their taking AP Calculus.

Over 400,000 high school students took an AP Calculus exam in 2014–2015, and the number increases between 5% and 6% each year.[12] Should we be happy about this?

We should first ask what our objectives are in encouraging more students to take AP Calculus, and then evaluate whether our objectives are being met.

Two generations ago, essentially no one (including the first author) took calculus in high school; indeed, most students began college with "analytic geometry," a topic which is now incorporated into high school precalculus courses. AP Calculus was introduced in order to provide the very best math students with "advanced placement," a way to start their college careers a semester or two ahead and get to more advanced mathematics more quickly.

However, for a variety of reasons, providing this option for a few students has, over the past 35 years, been transformed into a policy of accelerating the curriculum, including pushing Algebra 1 into 8th grade, so that large numbers of students can take calculus in high school and take advantage of this "advanced placement" option.

One question that should therefore be asked is, What percentage of the students who take AP Calculus actually take advantage of their acceleration by gaining and making use of advanced placement? That is, of those who take AP Calculus what percentage begin college with second-semester (or third-semester) calculus and remain one semester (or two semesters) ahead of students who did not take AP Calculus?

More recently, AP Calculus has been seen as a vehicle to significantly increase the number of students going into what is called the STEM pipeline. A 2007 report by the National Academy of Sciences, "Rising Above the Gathering Storm," sees offering more AP Calculus courses in high school as a major way of ad-

10 *Advanced Placement* and *AP* are registered trademarks of the College Board. For simplicity, these designations appear in this publication without the registered trademark symbol ®.

11 "The Rush to Calculus" can be found at dimacs.rutgers.edu/~joer/The-Rush-to-Calculus.pdf and "Why Do Students Rush to Calculus?" can be found at dimacs.rutgers.edu/~joer/Why-Rush-to-Calculus.pdf.

12 This information is from the College Board (collegeboard.com) which oversees the Advanced Placement Program.

dressing the country's pipeline problem and recommends training large numbers of high school math teachers to teach AP Calculus. This recommendation is based on the unsubstantiated assumption that students are more likely to pursue STEM careers if they take AP Calculus in high school than if they take calculus the following year in college. A more cost-effective solution seems to be to utilize the experienced calculus teachers in the nation's colleges to teach calculus, and to use the limited resources currently available (or any resources that are newly generated) for the professional development of high school teachers to improve their instruction in the prerequisites of calculus.

A second question that should therefore be asked is, Does taking AP Calculus in high school encourage students to pursue STEM careers in college?

Two studies conducted by the authors help shed light on these two questions and address the initial question of whether the nation actually benefits when more and more students are taking AP Calculus in high school.

In the first study, conducted in 2006–2007, the first author found that only a tiny percent (5.4%) of Rutgers College students who took a full-year calculus course in high school continued their acceleration through their first year of college—that is, took the next two math courses in their first year at Rutgers.

The other 94.6% of the students who took calculus in high school essentially used their years of acceleration in order to slow down (or even end) their mathematical studies.

Arriving at this conclusion involved substantiating four conjectures that the first author had formed some years earlier in a modest experiment:

1. about half of the Rutgers students who take calculus in high school take AP Calculus,
2. about half of those who take the AP Calculus course take the AP Calculus exam,[13]
3. about half of those who take the AP Calculus exam receive grades of 4 or 5 (sufficient to get college credit at Rutgers and comparable institutions), and
4. about half of those who receive grades of 4 or 5 actually continue on an accelerated path, that is, take the next two calculus courses in their first year at Rutgers.

Assuming that these conjectures are correct, we are led to the conclusion that only about 1 out of 16 Rutgers students (6.25%) who are accelerated into a calculus course in high school earn and make use of advanced placement in mathematics. To what extent this is true at other institutions is not addressed in this article.

With the assistance of the Rutgers Office of Institutional Research (OIR)[14], a sample of 400 students was randomly selected from the 2130 students entering Rutgers College in the fall of 2003, and Rosenstein reviewed all of their high school transcripts to see whether the first conjecture was correct. High school transcripts do not indicate whether students take the AP Calculus exam, but Rutgers OIR provided a list of all 332 Rutgers College students who took the AP Calculus exam and their scores, so to test whether the second conjecture was correct, Rosenstein compared the list of students in the sample who took the AP Calculus course in high school with the list of students whose scores on the AP Calculus exam were reported to Rutgers. He also used the second list to test the third conjecture about the scores of students on the AP Calculus exam. Finally, Rutgers OIR provided the math courses taken by all of the students in the second list, together with their grades, in their first two years of college, so that the fourth conjecture could also be tested.

The results were as follows:

1. 123 out of 217 students (56.7%) who took a full-year calculus course in high school took an AP Calculus course.

13 The first two conjectures may be correct for Rutgers students, but are apparently not true nationally. The first two conjectures imply that about one quarter of Rutgers students who take calculus in high school take an AP Calculus exam. However, from NCES data, between 750,000 and 800,000 high school students study calculus each year, while from College Board data, over 400,000 students take an AP Calculus exam, which implies that about one half of students who take calculus in high school take an AP Calculus exam.

14 We would like to thank Tina Grycenkov of the Rutgers Office of Institutional Research for helping gain approval and cooperation for conducting this study, and for gathering and facilitating our use of the needed information.

2. 59 out of 123 students (48.0%) who took an AP Calculus course took the AP Calculus exam.
3. 162 out of 332 students (48.8%) who took the AP Calculus exam received advanced placement (either for one semester, if they scored 4 or 5 on the AB exam, or for two semesters, if they scored 4 or 5 on the BC exam).[15]
4. 66 out of the 162 students (40.1%) who received advanced placement continued with their acceleration (that is, took the next two math courses in their first year at Rutgers).

This enables us to conclude that the percentage of students who took a full-year calculus course in high school and then continued with their acceleration through their first year at Rutgers is the product of these two fractions, that is, (59/217) × (66/332), or 5.4%. Even when we look only at the students who took an AP Calculus course in high school, the percentage of those students who continue with their acceleration through their first year at Rutgers was only (59/123) × (66/332) or 9.5%.

Thus a very small percentage of the students who were mathematically accelerated throughout high school actually continued their acceleration in college and actually took advantage of that acceleration. This data suggests that only 1 of 10 students currently taking AP Calculus actually takes advantage of the "advanced placement" that taking AP Calculus makes possible.

There are other reasons why students take AP Calculus, but schools and districts should know that only a small percentage of students who take AP Calculus gain and take advantage of the "advancement placement" that it promises. They should therefore carefully consider the consequences, both for the entire student body and for the entire curriculum, of encouraging 10 times that many students to take AP Calculus.

On the other hand, one might argue that taking AP Calculus benefits students even if they do not earn advanced placement and continue with their acceleration. To examine this argument, we considered four groups of students whose acceleration was interrupted and examined the data to see whether it sheds light on what happened to these students in college. The remaining 94.6% of the students can be partitioned into four categories, those who

a. Received a 4 or 5 on the AP Calculus exam but did not continue their acceleration.
b. Took the AP Calculus exam, but received a score of 3 or less.
c. Took the AP Calculus course, but didn't take the AP Calculus exam.
d. Took a full-year calculus course, but didn't take the AP Calculus course.

As to category (a), we noted above that 66 out of 162 students (40.1%) of the students who received advanced placement continued their acceleration. What happened to the other 96 students who didn't continue their acceleration?

- 52 took and successfully completed the next math course in their first semester at Rutgers and then took no additional math courses
- 17 took the next math course in their first semester at Rutgers, but received poor grades—that is, D, F, or W—and then took no additional math courses
- 17 took a semester off—that is, they took no math in one of their first two semesters at Rutgers, but then resumed the calculus sequence with the next calculus course
- 3 rejected advanced placement and started college with Calculus 1
- Of the remaining 7 students, 1 took no math courses, 1 took a math course for liberal arts students, and 5 had partial success in their math courses

A major benefit received by these students from taking AP Calculus is that it enabled 52 of them to stop taking mathematics a semester earlier. Can you imagine that—they were accelerated for four years so that they could get done with mathematics a semester earlier!

15 There are two Advanced Placement calculus exams, known as the AB exam and the BC exam. Students who take the high school equivalent of a one-semester college calculus course take the AB exam, whereas those who take the equivalent of a two-semester college calculus course take the BC exam; approximately 60% of the BC exam addresses topics on the AB exam.

On the other hand, it seems that 22 of these students (17 + 5) were not properly prepared by the AP Calculus course for their next math course, although these students' lack of success may have been due to other factors in some cases.

As to category (b), we noted above that 162 out of 332 students (48.8%) who took the AP Calculus exam received advanced placement. What happened to the other 170 students who did not receive advanced placement?

Although all of these students had on their high school transcripts the prerequisites for a first-semester calculus course, entry into Calculus 1 at Rutgers is permitted only to those who score sufficiently high on a reliable calculus-readiness placement test that has been administered by the Rutgers department of mathematics for over 30 years.

Of the 170 students who did not receive advanced placement, 131 placed into Calculus 1, but 30 were required to take a prerequisite course—precalculus or even Algebra 2; that is, 30 of these students were not ready for calculus even though they took an AP Calculus course in high school. One can plausibly claim that these students were disadvantaged by being permitted or even encouraged by their schools to take AP Calculus in high school; they would more likely have been better prepared for calculus in college if their high school courses were more focused on its prerequisites.[16]

Like most colleges and universities, Rutgers has two calculus tracks. One track begins with a "mainstream" course that leads to upper division courses in the mathematical sciences and that is taken by students majoring in math, physical sciences, and engineering; the AP Calculus AB course is intended to be comparable to the first semester of this track and the AP Calculus BC course in intended to be comparable to the first two semesters of this track. The other track is a two-semester "terminal" course, one or both semesters of which are typically taken by biology, business, economics, and psychology majors.

Of the 131 students who were placed into Calculus 1, 40 began the mainstream track and 91 the non-mainstream track. It is striking that the first college mathematics course taken by all but 40 of the 170 students who had taken the AP Calculus exam but were unsuccessful in gaining advanced placement was a lower level course than the AP Calculus course that they took in high school.

It is true, however, that almost all of the 131 students who took Calculus 1 were successful in their respective courses, all but 5 receiving grades of C or better and all but 35 receiving grades of B or better. However, this is not an argument for acceleration, since it is equally true, for example, that students who repeat Algebra II would learn it better than those who take it just once.[17] Moreover, in theory it is not unexpected that after completing what is advertised as a higher level calculus course, students will do well on a lower level course; however, since calculus in college is different from calculus in high school and, in general, being in college is different from being in high school, their relative success is not to be taken for granted. Finally, it should be noted that only 52 of these 131 students continued on to Calculus 2; the other 79 students successfully completed only one semester of calculus.

As to category (c), our sample of 400 transcripts included 64 students who took the AP Calculus course but did not take the AP Calculus exam. What is most striking about this group is that 20 of these students took no math course in their first two years at college; while there are many reasons that these students did not take any math for two years, it seems that AP Calculus functioned for them as a deterrent from math. Another 10 were placed into lower level courses. Only 33 took calculus, 15 the mainstream course, and 18 the terminal course. Most of them were successful in this course, all but 4 receiving grades of C or better. However, only 12 took Calculus 2 and only 7 of those received a grade of C or better. Once again, we can

16 Sadler and Sonnert provide evidence that taking calculus in high school improves calculus performance in college, regardless of preparation. However, that does not apply to these 30 students who were placed in precalculus. We cannot be sure that taking AP Calculus disadvantaged them, but it is certainly correct that they were disadvantaged by "being permitted or encouraged by their schools" to take it.

17 It may be that, in some cases, high school students' fear of calculus in college is a major reason why they choose to take calculus in high school; they may believe that the only way to succeed in college calculus is to have first seen the material in high school. For such students, acceleration provides them this opportunity, even though, as a result, they may be less well prepared for college calculus.

say that only a small percentage of students in this category, 7 out of 64, may have benefited from taking AP Calculus in that it prepared them for both Calculus 1 and Calculus 2 in college.

As to category (d), our sample of 400 transcripts included 94 students who took a full-year calculus course in high school other than AP Calculus. Only 6 of the 94 (6.4%) took no math courses in college, a far lower percentage than the 20 out of 125 (16.3%) students who took the AP Calculus course. It is hard to avoid the conclusion that taking AP Calculus served as a deterrent to mathematics for many more students than non-AP calculus; in any case, these data should raise legitimate concerns about the impact of proliferating AP Calculus courses.

A higher percentage of these students took Calculus 1 than the preceding group – those who took the AP Calculus but not the exam — (68 out of 94, or 72.3% compared to 37 out of 64, or 57.8%) and a higher percentage successfully completed Calculus 2 (15 out of 94, or 16.0%, compared to 7 out of 64, or 10.9%).

From all of these data it seems that the preponderance of the students who took AP Calculus but did not receive advanced placement would have been better served by a non-AP Calculus course.

Why then do so many students take AP Calculus in high school? That question was one of the primary motivations for the second study that was conducted in the spring of 2011 by both authors.

In the second study, we conducted an online survey in 2011 with the cohort of undergraduate students who entered the School of Arts and Sciences of Rutgers University, New Brunswick in the fall of 2007 (and were therefore about to graduate) and whose scores on the AP Calculus exam had been reported to Rutgers.[18] In the survey we asked them a number of questions related to their reasons for taking AP Calculus, their undergraduate programs, and their intended careers.

The Office of Institutional Research (OIR) at Rutgers provided us with the list of students in this cohort and their email addresses; OIR also provided us with the students' AP Calculus scores and a list of all the college math courses that they had taken over the past four years and their grades in these courses.

The 478 students on the list who were still at Rutgers were invited to participate in an online survey of about 25 questions; they were given a week to respond to the survey and those that had not yet completed the survey received two reminders during that week to do so. To provide the students incentive to participate in the survey, those completing every question in the survey were to be entered into a raffle for an iPad 2. Altogether 194 of the 478 students actually completed the survey; the high participation rate (40.6%) was presumably due, at least in part, to the popular incentive we provided.

In the questionnaire, we asked students to provide the following information, and we report on some of their responses in this abbreviated article.

- Why they took the AP Calculus course and exam.
- Whether taking AP Calculus increased their interest in math and/or encouraged them to pursue studies leading to a career in math, science, or engineering.
- What math course they actually took in their first semester in college.
- Why they were unsuccessful in getting AP credit (if that was the case).
- How many math courses they took in college.
- Whether taking math courses in college increased their interest in math and/or encouraged them to pursue careers in math, science, or engineering.
- What were their majors and what were their intended careers.
- Whether they thought that they had benefited from taking AP Calculus in high school.

To analyze the survey responses, we divided the participants into three groups:

A. Group A consisted of the students who had earned credit for both Calculus 1 and 2 (a score of 4 or 5 on the BC exam),

B. Group B consisted of the students that received credit for Calculus 1 (a score of 4 or 5 on the AB exam or on the AB portion of the BC exam) but not for Calculus 2, and

18 The survey did not include students in the College of Engineering. Performance of engineering students on the AP Calculus tests is discussed in a later footnote.

C. Group C consisted of the students that did not receive college credit for Calculus 1 (a score of 3 or below on the AB exam).

Of the 194 students who completed the survey, 37 (19.1%) were in Group A, 58 (29.9%) were in Group B, and 99 (51.0%) were in Group C. Thus in this cohort, the percentage of students taking the AP Calculus exam who received advanced placement was 49.0%, in accordance with the third conjecture addressed in the first study.[19] [20]

To learn why students took AP Calculus in high school, we provided the students with fifteen reasons and asked them to rate each statement according to five possible choices: strongly agree, agree, somewhat agree, disagree, and strongly disagree.

In the following table, the entries are the percentage of students in each of the three groups who agreed or strongly agreed with the reason given. (Note that in the survey itself, the statements were not organized into the italicized categories, nor were they presented in the order that they appear in the table.)

Why did students take AP Calculus in high school? There is clearly no simple answer. At least two-thirds of Group A students agreed or strongly agreed with ten of the fifteen reasons in the survey, and at least two-thirds of each of Group B and Group C students agreed with seven of the fifteen reasons.

Five reasons appeared on all three lists — that is, in all three groups, two-thirds of the students responded most positively to:

- I really liked math when I was in high school
- I wanted to start off in college with a higher level course
- I wanted to be better prepared for college courses
- AP Calculus looks good on college applications
- My teachers or counselors suggested that I take it.

If we look instead at the reasons with which students "strongly agreed," we find that the top five reasons for students in Group A were

- I really liked math when I was in high school (83.8%)
- I wanted to start off in college with a higher level course (70.3%)
- I enjoy challenging math courses (64.9%)
- AP Calculus looks good on college applications (56.8%)
- I had to take some math course in my senior year (54.1%)

Whereas 20 of the 37 students (54.1%) in Group A gave the fifth response above, 19 students each "strongly agreed" that "I wanted to learn more higher level mathematics," "I planned to major in a math related subject, "my friends were taking AP Calculus," and "it was expected of me."

The top five reasons for students in Group B were

- I wanted to start off in college with a higher level course (58.6%)
- I really liked math when I was in high school (51.7%)
- AP Calculus looks good on college applications (51.7%)
- I wanted to be better prepared for college courses (48.3%)
- I wanted to learn more high level mathematics (46.6%)

The top five reasons for students in Group C were

- AP Calculus looks good on college applications (48.5%)
- I wanted to start off in college with a higher level course (44.4%)

19 Actually, of all 478 students who took the AP Calculus exam, only 193, or 40.4%, received advanced placement; 13.4% would have been in Group A (if they had completed the survey) and 27.0% in Group B. If all 478 students had completed the survey, then 59.6% would have been in Group C. It is not surprising that the students who were more successful in AP Calculus were a bit more likely to participate in a survey related to AP Calculus than those who were unsuccessful.

20 Of the 216 students in the College of Engineering, 134 received advanced placement. If they are combined with the students in the College of Arts and Sciences, 327 out of 694, or 47.1%, received advanced placement, which is still less than half of the students who took the AP test.

Statement	Group A ($N = 37$)	Group B ($N = 58$)	Group C ($N = 99$)
Intrinsic rationales			
I really liked math when I was in high school	94.6	89.6	67.7
I wanted to learn more higher level mathematics	83.8	79.4	45.4
I enjoy challenging math courses	89.2	79.0	52.0
Educational rationales			
I wanted to start off in college with a higher level course	83.8	84.5	74.7
I wanted to be better prepared for college courses	78.4	74.2	72.7
I planned to major in a math related subject	67.6	45.6	25.2
Pragmatic rationales			
AP Calculus looks good on college applications	78.4	82.7	80.8
Taking AP courses would enable me to save money by graduating from college in three years	21.6	20.7	24.2
Taking AP course would boost my GPA	45.9	32.8	30.3
Social rationales			
My friends were taking AP Calculus	81.1	53.5	69.7
My parent wanted me to take AP Calculus	51.4	39.7	46.5
It was expected of me	78.4	62.1	61.2
Default rationales			
I had to take some math course in my senior year	62.2	48.3	70.6
My math teachers or counselors suggested that I should take it	73.0	69.0	75.5
Negative rationales			
I wanted to avoid taking math in college	5.4	5.1	30.3

- I had to take some math course in my senior year (42.9%)
- I really liked math when I was in high school (41.4%)
- I wanted to be better prepared for college courses (39.4%)

What stands out in these data are:

- The difference between the enthusiastic response of Group A students to the "intrinsic" rationales for taking AP Calculus and the lukewarm response to those rationales among Group B and Group C students.
- The belief that AP Calculus looks good on college applications and the concomitant, though often undeclared, encouragement of math teachers and counselors, plays a prominent motivational role in

students' taking AP Calculus. These factors seem to play a greater role than pressure from parents or peers.

- The percentage of students who "strongly agreed" to any of the reasons decreased substantially from Group A to Group B to Group C.
- Not surprisingly, the students with the strongest performance on the AP Calculus exam (Group A) had the most positive attitudes towards mathematics. They really liked math in high school, wanted to start off college with higher-level math course, wanted to learn higher-level math, and enjoyed challenging math courses.
- Approximately 71% of Group C agreed or strongly agreed that they took AP calculus because they had to take some math in their senior year and 30% of Group C students who took the AP Calculus course wanted to avoid taking math in college.
- While a high percentage (25 out of 37, or 67.6%) of group A students planned to major in a math related subject in college, fewer than half (26 out of 58, or 45.6%) in Group B and barely a quarter (25 out of 99, or 25.2%) in Group C planned to major in a math related subject. Thus a substantial percentage of the students in groups B and C took AP calculus even though they did not plan to major in a math related subject and were, therefore, not seeking to be "advanced" in mathematics.
- On the other hand, it is interesting that the number of students in each of the three groups who agreed or strongly agreed that they "planned to major in a math-related subject" was 25 or 26; that is, the potential number of math-related majors from each of the three groups is about the same, although the three groups are of different sizes.
- In all three groups more than half of the students said that it was expected of them to take AP Calculus in high school. It is striking that so many students felt that they were "expected" to take the "advanced" math course. The exception has become the norm; that is, whereas AP Calculus was originally introduced to permit the best students to obtain "advanced placement" in mathematics, it was now expected that students should routinely take AP Calculus.
- None of the three groups thought that saving money (by spending fewer years in college) was an important reason for taking the course and fewer than half of the students in each group thought that taking the course would boost their GPA.

It is clear from looking at the observations above that the three groups had very different reasons for taking AP calculus in high school. While students in group A were more inclined to take higher level math courses and pursue a math related major, many students in group C wanted to be done with math quickly and thought that AP Calculus would help them achieve that goal.

In the survey we also asked students why they took the AP Calculus exam, since in the previous study we had learned that only about half of the students who take the AP Calculus course take the AP Calculus exam. We offered students ten possible reasons for taking the AP Calculus exam and offered the same options as possible responses. In this table also, the entries are the percentage of students in each of the three groups who agreed or strongly agreed with the reason given.

Why did students take the AP Calculus exam? Again the answer is not clear and differs among the three groups.

- About 80% of the students in groups A and B (group A more "strongly") wanted to start college with Calc 2 and Calc 3 but very few students in group C wanted to do the same. This reflects, again, that many students in group C were not looking for an "advanced placement" in math when they reached college.
- As expected, group A students enjoyed studying math and taking math exams the most, followed by Group B and then Group C.
- Group A students were more likely than the other groups to agree that "Colleges give extra consideration to applicants who are planning to take AP exams." It is not clear that this belief corresponds to reality since colleges make decisions about applications before any AP exam scores are received.

Statement	Group A ($N = 37$)	Group B ($N = 58$)	Group C ($N = 99$)
Intrinsic rationales			
I enjoyed studying math and taking math exams	86.5	67.3	43.4
Educational rationales			
I wanted to start off at college with Calc 2 or Calc 3	78.4	81.1	29.3
I thought that I would learn the course material better if I planned to take the exam	59.4	41.4	52.5
Pragmatic rationales			
Colleges give extra consideration to applicants who are planning to take AP exams	56.7	48.3	41.4
Taking the AP exam and getting advanced placement would enable me to save money by graduating from college in three years	21.6	22.4	25.3
Social rationales			
My friends were taking the AP Calculus exam	62.1	48.3	48.4
My parents wanted me to take the AP Calculus Ex-am	54.0	41.3	44.5
Default rationales			
My math teachers or counselors suggested that I should take it	67.5	68.9	70.7
Negative rationales			
I wanted to take as few math courses as possible in college	21.6	13.8	45.4
Requirements			
All students in AP Calculus were required to take the exam	29.7	31.0	39.4

- Significantly more students in group C wanted to “take as few math courses as possible in college.” This reflects that Group C students were less inclined towards a math-intensive major and perhaps viewed AP Calculus as a means to avoid future math courses. It is striking that, although 30.3% gave this as a reason for taking the AP course in the previous question, this percentage rose by half to total 45.4% when the students were asked to explain why they took the AP Calculus exam. Although a number of factors may have served to discourage these 15 students from continuing to take math courses, it is reasonable to conjecture that the AP course itself played a significant role.
    - o What conclusion can be drawn from the data that 45 out of the 99 students in group C want to “take as few math courses as possible in college”? Although there can be many reasons why these students wanted to avoid math courses in college, it is not unreasonable to use these data to confirm the conjecture in the previous study that “the acceleration strategy has produced a lot of negative attitudes about mathematics.”

- The strongest reason by far for students in Group C to take the AP Calculus exam is that "my math teachers or counselors suggested that I take it." Although a similar percentage of students in Groups A and B agreed or strongly agreed with that reason, in both of those groups the first two reasons and, in particular, the intention to start off at college with Calc 2 or Calc 3 was more prominent.
- If we compare the responses in the second table to the statement "I wanted to start off at college with Calc 2 or Calc 3" with the responses in the first table to the comparable statement "I wanted to start off in college with a higher-level course," we find a major difference between both Groups A and B on the one hand and Group C on the other. About 80% of the students in both Groups A and B responded "agree" or "strongly agree" to both of those statements. However, although 72.7% of the students in Group C responded "agree" or "strongly agree" to the first statement (as to why they took the AP course), only 29.3% responded "agree" or "strongly agree" to the second statement (as to why they took the AP exam). That is, after taking the AP course, about 60% of the Group C students who intended to start college with a higher-level math course had lowered their expectations so that by the end of the school year, when the AP calculus exam was given, they no longer expected that they would earn advanced placement.

We asked students to describe the extent to which taking AP calculus and its prerequisites in high school encouraged or discouraged them from pursuing studies leading to a career in math, science, or engineering; they had to choose one of five options: "They encouraged me a great deal toward such careers," "They encouraged me somewhat toward such careers," "Taking those courses didn't make much of a difference," "They discouraged me somewhat from such careers," and "They discouraged me a great deal from such careers." Students were offered the opportunity to provide a written explanation of their response to this question and a substantial number (139 out of 194, or 71.3%) responded.

- About two-thirds of the students in both groups A and B responded that taking the AP Calculus course encouraged them toward careers in math, science, or engineering, and about one-third indicated that taking these courses made no difference. These fractions are reversed for Group C; that is, only one-third said that taking the AP Calculus course encouraged them toward careers in math, science, or engineering, and two-thirds indicated that taking these courses made no difference or discouraged them from such careers.
- For a great number of students in this survey (approximately one-third in groups A and B and approximately half in group C) taking the AP Calculus course made no difference toward their decisions to follow a STEM career path.
- Although many students indicated that taking the AP Calculus course reinforced their intentions to become STEM majors, there were essentially no students who indicated in their comments that they planned to become STEM majors because they took the AP Calculus course. This finding contradicts the assumption of "Rising Above the Gathering Storm" that taking AP Calculus serves as a means to expand the STEM pipeline.
- Almost 20% of the students in group C, who had taken a year of AP Calculus, were considered by Rutgers as being unprepared to take Calculus 1. Overall, 12% of all students (as group C made up 60% of this cohort) who took the AP exam were unprepared for calculus at Rutgers even after taking a full-year AP Calculus course in high school. These students were not prepared for AP Calculus in the first place, and should have been steered away from AP Calculus.

As in the previous study, we examined the math courses taken by the students at college and their grades. Although for the first study Rosenstein only had data on math courses taken in the first two years, for the present study we had data on courses and grades for four years. For the present study, we also knew the subjects in which students had majored, since we asked them this question on the survey.

- In group A, 86.4% were majors in math, science, or engineering, and 70.3% were majors in math-intensive subjects (including the physical sciences) that required a number of math courses.

- In group B, 75.9% were majors in math, science, or engineering and 43.1% majored in math-intensive subjects. A high percentage of group B students were inclined towards the biological sciences that required fewer math courses.
- In contrast, among students in group C, only 35.4%, were majors in math, science, or engineering and only 11.1% had math-intensive majors; there was a sharp decline from the number of students who intended to major in a math-related subject to the number who actually did. One might conjecture that the percentage of these students who were put off from math by their experience in the AP Calculus course was not insignificant.

In the survey, we asked students the "Yes-No" question "Are you planning a career in math, science, engineering, or technology?" and we also asked them to tell us what career they were planning to pursue. 174 students answered the second question, and the remaining 22 wrote "undecided" or did not tell us what career they were planning to pursue — thus 52/174 in the bottom cell at the right means 52 of the 174 who actually responded with their intended career. The results are summarized in the following table:

	Group A ($N = 37$)	**Group B** ($N = 58$)	**Group C** ($N = 99$)	**Total** ($N = 194$)
Intends career in math, science, engineering	31 (83.8%)	47 (81.0%)	46 (46.5%)	124 (63.9%)
Listed career in math, science, engineering	26 70.3%)	41/52 (78.8%)	25/83 (30.1%)	92/174 (52.9%)
Listed career in math-intensive subject	20 (54.1%)	20/52 (38.5%)	12/83 (14.5%)	52/174 (29.9%)

Summary of intended careers of all students in the study:

- There is clearly a discrepancy between what the students regard as "math, science, engineering, or technology" and what is commonly understood as falling under these categories. On the other hand, there seems no general agreement as to what falls under these categories. For example, there does not seem to be general agreement as to whether "physician" is a STEM career.
- The discrepancy between the percentage of group C students who indicated that they were pursuing careers in math, science, or engineering (first row of table) and the percentage of group C students who listed careers in math, science, or engineering (second row of the table) can be explained by observing that the health-related careers that many students indicated they were pursuing are usually not considered careers in math, science, or engineering (or STEM careers) since the science requirements for those careers are often very modest.
- Only ten of the 194 students intend to be high school teachers (seven math and three science), areas in which the United States has a critical shortage.[21] Only a handful indicated that they intend to be engineers, but that is not surprising since the survey involved only students in the School of Arts and Sciences, and not those in the School of Engineering.
- These data do not provide any evidence that increasing the number of students taking AP Calculus is having the desired effect of increasing the number of students intending to teach in high school.

In the survey we asked students whether they benefitted from taking AP Calculus in high school and whether they would recommend it for future students, and most agreed with those conclusions. However, some of the students cited AP Calculus as a vehicle that allowed them to take fewer math courses at college and avoid experiencing bad teaching at the college level.

21 According to an article by Christopher Drew in the *New York Times* of 11/4/11 entitled "Why Science Majors Change Their Minds": "The president and industry groups have called on colleges to graduate 10,000 more engineers a year and 100,000 new teachers with majors in STEM—science, technology, engineering and math."

Recommendations

The two most important findings of these studies are that, first, a very small percentage of those who are accelerated throughout high school maintain that acceleration through their first year at college and, second, that there is no evidence that encouraging more students to take AP Calculus will expand the STEM pipeline.[22]

Yet the curriculum in many schools is organized to accelerate students into AP Calculus—beginning with the policy of expecting more (or even, all) students to take Algebra 1 in the 8th grade—often taught by middle school teachers who lack the appropriate mathematical background or credentials and who provide their students with a substandard introduction to algebra.

Moreover, more and more students are encouraged to take AP Calculus, including those students who struggled to complete Algebra 2, Geometry, and Precalculus in the 9th, 10th, and 11th grades.

The driving force behind the practice, as seen from the students' responses and the recommendations of their teachers and counselors, is the broad perception—perhaps correct, perhaps incorrect—among parents and school personnel that colleges routinely favor students for admission if they have AP Calculus on their transcripts.

The basic recommendation is "STOP these practices."

Many detailed recommendations follow. But none of these will be implemented until colleges change their admissions policies and practices—that is, they should not "routinely" favor students with AP Calculus —or, if that is already the case, they make it clear to students, parents, and counselors that this is not their practice.

Many students, teachers, counselors, and principals believe that taking AP Calculus enhances the possibility of admission to college. As a result, many students take AP Calculus who do not benefit from that course. Ideally, colleges should announce that an application for admission will only be enhanced by taking AP courses if the students take the exams and score 4 or 5 on them. Unfortunately, that is impossible to enforce since the students are typically admitted to college well before the AP exams are given. However, we encourage colleges to adopt a policy of the following type in evaluating the AP credentials of applicants for admission.

Recommendations to Colleges on Admission Policy

- With each student's application to admission to college, the high school must submit a document indicating, for each of the previous five years, the number of students who took an AP Calculus course, the number of students who took the AP Calculus exam, and the distribution of scores of those students on the AP Calculus exam.[23]
- Each college should modify its admissions policy so that it gives extra weight for taking the AP Calculus course only to those students whose high schools can report that they treat the course seriously, not just as a means for enhancing college admissions. For example, a college might decide to give extra weight if 90% of the students taking the AP Calculus course in that high school took the AP Calculus exam, and that 75% of the students who took the exam received a score of 4 or 5.
- It might be challenging for each individual college to determine whether a high school treats their AP Calculus course seriously, and it would therefore be recommended that the College Board use its own database to provide guidelines to colleges.

Recommendations on curriculum and entry to AP Calculus and non-AP calculus courses:

22 An important exception must be noted. There are many schools in the nation (perhaps even in New Jersey) in which AP Calculus is not offered at all, usually schools in high-poverty urban and rural areas, and we certainly lose from the STEM pipeline those students from such schools who, had they had the opportunity to take AP Calculus, would belong to groups A and B.

23 Providing such data would only make sense for high schools that have a substantial AP Calculus population and, in that case, privacy concerns would be avoided since all data requested is aggregate.

- Students should take Algebra 1 in 8th grade only if they have teachers who have the appropriate background and certification in teaching mathematics.[24]
- Students should be discouraged from taking any course if they did not achieve at least a B (and ideally a B+) in the previous course.[25]
- Students should be encouraged to take an AP Calculus course only if they
  - o received an A or B+ in the Precalculus course,
  - o indicate that they like mathematics and wish to be challenged,
  - o intend to take more advanced math courses in their first year in college, and
  - o are either mathematically talented or are strongly considering a career that requires a substantial number of college mathematics courses—i.e., mathematics, statistics, and the physical sciences.[26]

Otherwise, AP Calculus is not the ideal course for them, and teachers and advisors should discourage them from taking AP Calculus.

- School counselors should receive training that will enable them to identify students who should take the AP Calculus course and to actively and convincingly discourage students who should not be taking AP Calculus. This includes
  - o students who are not prepared for AP Calculus—recall that at least 18% of the students who had completed a full-year AP Calculus course were placed at Rutgers into Intermediate Algebra or Precalculus
  - o students who insist that although unprepared for AP Calculus, they will succeed in calculus because they will seek extra help or private tutoring to keep up with the material in the course
  - o students who are not mathematically talented and are intending to major in a subject that has limited mathematical prerequisites (including biology)—recall that only 76 of the 194 students agreed or strongly agreed that they took the AP Calculus course as "I planned to major in a math related subject."
- Students and parents should be made aware that the AP Calculus course is challenging, intended for math-intensive majors and not necessarily going to lead to an advanced placement in mathematics or save them time in college.
- All students who take the AP Calculus course should be required to take the AP exam.
- AP Calculus should not be the "default" mathematics option for seniors; that is, the course that they take automatically after they complete Precalculus.[27] Since all college-bound students should take a math course in their senior year of high school, schools should design and offer other attractive and

24 The transcripts in the first study show that although Algebra 1 in the 8th grade benefits the best students by smoothing their path to AP Calculus, there is no evidence that it benefits average students. That is, for average students, the Rutgers course-taking patterns of those who took Algebra 1 in the 8th grade was not significantly different from those who didn't take Algebra 1 until the 9th grade. The most striking difference between the two groups is that 32 of the students who took Algebra 1 in the 8th grade did not take any mathematics course in their first two years at Rutgers.

25 If parents insist on their child's taking the next course, as they may be able to do in many locations, or if the district does not have the resources to provide a "next course" that incorporates review of the previous course, then the district should provide supplementary instruction based on diagnostic assessments of the student's understanding of and ability to use the mathematics addressed in the previous course.

26 One has to question on an individual level why many group C students took the AP course when they already knew that they were not mathematically talented and that they were not going to be a STEM major down the road. Also, one has to ponder at a policy level if it is it worthwhile to spend resources teaching calculus to students similar to those in group C; many of them would derive greater benefit from a course that provided an informal introduction to calculus, combined with probability, statistics, and discrete mathematics.

27 It appears that treating AP courses as "default" options is not unique to mathematics. Indeed, it seems that many students take several AP courses in their senior year because their high schools do not offer anything else. Although it is conceivable that it is appropriate for a student to take, simultaneously, AP courses in chemistry, mathematics, computer science, and English, it is rather unlikely that the student feels equally passionate about all four. It would make sense for high schools to require students to choose one subject (or at most two subjects) in which they take the AP courses.

interesting courses that seniors can take after completing Precalculus. These courses should include material that reinforces algebra and geometry skills, should include probability, statistics, and discrete mathematics, and should build on the Precalculus course (including some calculus topics) so that students completing these courses will be prepared for college math courses.[28]

In this study we found no evidence that increasing the number of students taking AP Calculus expands the STEM pipeline. The problem is not that there are too few students in the STEM pipeline, but that the pipeline is too leaky; that is, students who were previously attracted to math (and similarly for science) have decided not to pursue careers in those areas. The national focus should not be on recruitment, but on retention of students already in the STEM pipeline — this applies at all grade levels, but perhaps particularly at the college level[29 30 31] — and on providing alternate ways of entering the pipeline than AP Calculus — such as stronger and more appealing courses at the high school level.

But even at the high school level, there are important strategies for encouraging students to consider STEM careers:

- Implement appropriate and early interventions to cultivate and enhance younger students' interest in mathematics, science, and engineering. For example, students can be exposed to field trips related to STEM careers or be given an opportunity to talk to real scientists, mathematicians, and engineers to discuss the challenges and rewards of such careers.
- Interestingly, in all three groups in the second study a number of students claimed that the teachers were far more influential than the course itself in terms of motivating them to pursue or not pursue a STEM major. Teachers of science and mathematics, and particularly the teachers of AP Calculus, should be trained to discuss career options and real life applications of the ideas that students learn in the course.
- Ensure that students have positive experiences in their math classes by developing interesting courses with material that students see as relevant to their education, their careers, and their lives; that students get the assistance that they need to succeed in mathematics; and that they are not pushed ahead at a pace that will cause them to avoid mathematics in the future.

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28 How will schools and districts find the resources to provide such courses? The simple answer is by reducing the number of calculus courses. Of course, a teacher will have to learn to teach such a course, and will typically need to participate in some professional development, so these courses will require additional resources. But this has to be weighed against the substantial cost to the students and the community from placing students into inappropriate courses.

29 According to an article by Christopher Drew in the *New York Times* of 11/4/11 entitled "Why Science Majors Change Their Minds": "Studies have found that roughly 40 percent of students planning engineering and science majors end up switching to other subjects or failing to get any degree. That increases to as much as 60 percent when pre-medical students, who typically have the strongest SAT scores and high school science preparation, are included, according to new data from the University of California at Los Angeles."

30 On the other hand, the current economic situation may have had a major impact on the number of students who are pursuing STEM careers. David Bressoud reports, based on data from the UCLA's Higher Education Research Institute, that: "There has been a strong upward trend toward mathematics, the sciences, and engineering over the past decade. It has recently accelerated. In the past five years, the number of students intending to major in mathematics has risen by 31%, in the physical sciences by 37%, in engineering by 44%, and in the biological sciences by 67%. Most of this growth has occurred in just the past three years, since 2007. This is most dramatic within engineering, which went from 102,000 freshmen intending to major in this discipline in the Fall of 2007 to 156,000 in Fall 2010."

31 It should also be noted that a substantial percentage of the students who participated in this survey, between 84.2% and 91.8% of those who intended to major in math, science, or engineering actually did. This observation should not be taken to imply that taking AP Calculus results in staying in the STEM pipeline, but rather that those who intend to stay in the STEM pipeline typically take AP Calculus.

The Song Remains the Same, but the Singers Have Changed

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The introductory course in single variable calculus has changed only modestly in the last five decades. In 1967, when I first took Calculus 1, the only "real world" applications of integration were work and liquid pressure. Today, modern texts provide many more varied applications in examples and homework assignments. The only instructional practice in my class was the lecture, and the only assessments were three tests. While tests are still a favorite, today's classes may include laboratory activities, group projects, and online homework. Students may engage in inquiry-based learning, attend flipped classes, and explore concepts using dynamic software. Test questions are becoming more focused on concepts and less on techniques, though techniques still dominate as they should. Despite all the changes in instruction, the content and goals of a first year in mainstream calculus are very much the same today as they were nearly 50 years ago.

What subject other than mathematics can offer essentially the same course in both 1967 and 2016? Could anyone imagine teaching a "modernized" version of a 1967 course in biology (biology without recombinant DNA or protein folding) or chemistry (chemistry without fullerenes or computational chemistry) or Physics (physics without the charm quark or dark matter) or computer science (computer science without personal computers)? But mathematics is different from other subjects. Physicists can be wrong about the structure of the universe, and their theories can be updated and corrected. Where science has theories, mathematics has theorems; and theorems are not theories. We are not wrong about calculus, and our understanding of calculus doesn't change with time. So, yes, the core content of a 1967 calculus class can still be appropriate content today for students preparing for mathematics intensive majors and careers. As noted in Insights and Recommendations from the MAA National Study of College Calculus, "The content of Calculus I has remained relatively stable over the decades…" [Bressoud, Mesa, Rasmussen, 2015]

Calculus has not changed, but the calculus class has changed dramatically, because the student in the class has changed. Many of today's calculus students are not the students for whom the course was designed, and it shows.

Traditional Calculus Students

In 1967, essentially no one took a course in calculus who did not intend to take other core courses in applicable mathematics. Calculus was not an elective, and calculus students were a select group. For the most part, students were sitting in a calculus class because they were committed to a STEM major, and the course they took was created with that commitment in mind. The calculus class was populated with students who understood the importance of this mathematical tool so necessary for their success in their chosen fields. The primary (for many, the only) purpose in taking calculus was to get good at it for both later courses and for their intended profession. This is largely still true today in the colleges, where students rarely take calculus unless their degree program requires it. But it is not so true in the high schools. High school students take calculus for a variety of reasons, often unrelated to their career aspirations.

Today's Calculus Students are Non-Traditional

The rush to calculus, so well documented in David Bressoud's June 2015 *Launchings* column, *Calculus at Crisis II: The Rush to Calculus,* is real, and its reality has enormous consequences for both colleges and high schools. In the high schools, the pressure to reach calculus before graduation stems largely from the college admissions process. Beginning in earnest in the mid- 1980s the rush has been fueled by reports like the survey of admissions officers in the fall of 1991, which found that "Fifty-eight percent of the colleges in the sample reported that it had become progressively more difficult to be admitted without AP or honors coursework" [Rothschild, 1999]. For schools to have their top students be competitive for admission to selective schools, the need for AP courses was obvious, and for many schools, AP Calculus, principally Calculus AB, was one of easiest courses to offer, requiring only a single qualified teacher. After all, unlike many AP subjects whose content may be quite different from the course a teacher took in college 20 years earlier, calculus never changes, and so for schools with only a few AP offerings, AP Calculus AB is one very likely to be offered.

In order to have a group of students ready for calculus in their senior year of high school, some compacting of courses and content is required, often limiting the breadth and depth of preparation in mathematics courses leading up to calculus. Dan Kennedy, former Chair of the AP Calculus Test Development Committee, comments

> When you get right down to it, the pre-calculus rope comprises a surprisingly small number of algebraic and geometric strands, especially if you know you are preparing specifically for AB Calculus, which does not involve vectors, infinite series, or polar coordinates. Topics like statistics, probability, matrices, mathematical induction, graph theory, linear programming, and even financial topics like amortization and mortgages that will affect almost every student someday, are given short shrift in the core curriculum precisely because they are not necessary for studying calculus." He further states, "to summarize the teleological effect of AP on the high school curriculum, you have AP Calculus as an end driving a more focused pre-calculus preparation as a means, limiting the potential for horizontal enrichment in the earlier grades. [Kennedy, 2005]

A small study at Rutgers University by Rosenstein and Ahluwalia illustrates this core issue. In a survey of a random sample of Rutgers students taking calculus, they found that the largest percentage of students who reported that college admissions was the most important factor in their choice to take calculus in high school were in the group that, after a year of calculus in high school, were deemed not ready for the study of calculus and placed into precalculus or even college algebra. According to Rosenstein and Ahluwalia, "… more and more students are encouraged to take AP Calculus, including those students who struggled to complete Algebra 2, Geometry, and Precalculus in the 9th, 10th, and 11th grades. The driving force behind the practice, as seen from the students' responses and the recommendations of their teachers and counselors, is that colleges routinely favor students for admission if they have AP Calculus on their transcripts." [Rosenstein, Ahluwalia, n.d.]

The recent re-issue of the MAA/NCTM Joint Position on Calculus was initiated by requests from AP Calculus teachers to the MAA Governor-at-Large for High School Teachers for help in fending off parents and administrators who wanted to further streamline the preparatory program leading up to calculus. That struggle continues. Earlier this year, the following message was sent to the AP Calculus discussion group, "Our district has a goal to provide a pathway for students that will get them to AP Calculus (AB or BC) their junior year… We have also been told that precalculus can be skipped, so this is obviously a widespread belief—we are not buying it because of the deficit in skills this would produce for students taking Calc BC/Calc 2 in college." [AP Calculus discussion, 2015] Similar messages appear in the AP discussion forum several times each year, and the consistency and frequency of these messages illustrates the pressures on teachers to accelerate their students. The quickest and simplest way to do that is to reduce the preparatory material to just that content needed for success in Calculus AB.

The MAA/NCTM Joint Position on Calculus stresses the need for adequate preparation and the AP course description suggests four years of secondary mathematics and gives an explicit list of topics that are necessary for success:

> Before studying calculus, all students should complete four years of secondary mathematics designed for college-bound students: courses in which they study algebra, geometry, trigonometry, analytic geometry, and elementary functions. These functions include linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric, and piecewise-defined functions. In particular, before studying calculus, students must be familiar with the properties of functions, the algebra of functions, and the graphs of functions. Students must also understand the language of functions (domain and range, odd and even, periodic, symmetry, zeros, intercepts, and so on) and know the values of the trigonometric functions at the numbers $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, and their multiples. [College Board, 2012]

Nevertheless, many high schools are reducing the time allocated for the preparatory material and increasing the time allocated for calculus. Many schools teach Calculus AB to juniors followed by Calculus BC in the senior year. Of course, the extended time in calculus allows for the development of many of the precalculus concepts in the process of learning calculus, and it gets calculus on the students' transcripts in their junior year.

It is not uncommon for AP Calculus teachers to tout their students' strong performance when repeating the introductory course in college as a success story for high school calculus. Many high schools, recognizing some weaknesses of their students, have the expressed goal for calculus of preparing students for success in calculus when they repeat the course. It is becoming quite common to use a course in calculus as preparation for calculus, creating many of the well-documented problems for the college instructor. But what explains why so many of the students repeating calculus in college struggle in the course and why so many who took calculus in high school place into precalculus as a first course in college?

How Students Learn and How They Forget

There is a myriad of explanations and possible reasons for the difficulties of students moving from high school calculus to college calculus. The usual issues of the transition from a parent-directed household to living on campus, working a job to pay for school, sharing a dorm room on a hall with 20 other students each struggling with his or her own version of similar issues all play a role. But why does calculus seem to suffer more than other college courses? The easy explanation suggests that the high school course was deficient in some important way, and certainly the slower pace of the high school course does little to prepare students for the rapid pace of college calculus courses. Also, some teachers in high school are less well prepared than they should be, and, as has been well-documented and discussed in the *Launchings* columns, a weak precalculus preparation limits the students' capabilities with and understanding of the core ideas of calculus.

But there is another component to the high school setting that has been little discussed and which might play a significant role. Before students can learn calculus in a manner that has some significant residual, they must want to learn calculus. One of the major advances in brain science in the past decade has been a growing understanding of the different roles that a learner's goals and intentions play in the brain's construction of understanding and memory and how these affect the learner's ability to retrieve that information later. [Jee and Wiley, 2007; Kaplan, Damme, and Levine, 2012; Grèzes, Costes, and Decety, 1999] Our brains can learn well almost anything that we really want to learn and are prepared to learn. While still quite preliminary, this and similar research may help explain why students who do well in high school mathematics, even scoring well on the AP Calculus exam, may struggle in the post-calculus courses or when repeating calculus in college.

For the vast majority of high school students not intending to pursue mathematics-intensive career paths, their primary goal for calculus in high school is to pass the course. They may not consider mastery of the content essential to their long-term goals, and they can attend to the course as a means to a desired end. This is not unlike the manner in which students taking a college algebra or precalculus general education requirement course attend to the goal of passing that required course. While the students in high school taking calculus may be significantly stronger than those taking College Algebra, their goals for learning and their approach to learning can be quite similar. When the goal is not to develop a deep and abiding understanding and facility with the tools of calculus, but to pass the course with a good grade, either because the students do not value calculus as an important part of their career path or because they know they will be repeating calculus in college, the learning can be quite superficial. So, months after the course ends, the information has faded from memory, but their confidence in their ability to "relearn" calculus has not. And that's where the difference in the pace of college calculus can take its toll.

David Bressoud comments in an article for NCTM's *Math Teacher,*

> … just because students can succeed in calculus in the supportive environment of a high school does not guarantee that they will be successful when they get to college. The most useful skill for success in college is the ability to learn on one's own, to be able to think critically about what one reads or views in videos, and to use this critical analysis to build a personal, coherent, and functional mental structure for the many concepts of calculus. Getting students successfully through a test, even an AP Calculus exam, is by itself no guarantee that they will be successful in college. [Bressoud, 2015]

As a thought experiment, remember back to your general education requirement courses in college. Recall the difference in the level of attention, effort, and intellectual energy given to core courses in your intended major and that given to courses that were undesired but required to satisfy your graduation requirements. For many high school students, calculus, whether AP or not, is treated as a general education hurdle to be overcome or a box to be checked.

Changes on Both Sides

In addition to calling for a strong precalculus preparation before beginning the study of calculus, the MAA/NCTM Joint Position on Calculus charges colleges and universities to adapt to the changing student population entering freshman calculus courses. The strongest students from high school and those most motivated to continue their mathematical development for use in STEM disciplines will often use their AP credit to skip university calculus altogether, entering with advanced standing. But, as indicated in the MAA study, most of the students entering first semester calculus as freshmen have seen calculus in high school. And for some, "seen" is the correct verb when taken literally.

A number of colleges and universities have taken the MAA/NCTM charge to heart and have significantly modified their introductory calculus experience to accommodate the changing student profile. Macalester College has been a leader in both the extent to which the calculus program has been modified and in advertising their changes to others to serve as an exemplar for what is possible. Macalester now offers three courses in applied multivariable calculus suitable for students without prior calculus in high school and for those with AP Calculus experience. [Macalester Course Catalog, 2015]

The changes in the presentation of calculus required by the changing roles calculus plays in college admissions cannot be adequately addressed on only one side of the high school to college transition. The high schools and AP Calculus have roles to play as well, though exactly what those roles should be is unknown. It is, however, abundantly clear that the current structure is not working well for many students and institutions, and can be detrimental to some students' ability to continue their study of mathematics.

Calculus BC, the two semester high school equivalent to a two semester college calculus class, works well for those students for whom the course was originally created. These STEM-committed students intend to learn the material for future use and plan to use the credits or placement earned through the AP exam to continue their study of mathematics.

The content of Calculus AB, a course which enrolls more than three-fourths of the high school students taking AP Calculus, covers approximately 60% of the content of a typical two semesters of college calculus, but is taught over a school year. The pace is slower than that of the BC course and significantly slower than that of the first semester college course. But the slower pace offers a space in which change could happen. One possibility is to scale the core content back to that of a standard one semester course. This course would maintain the current level of rigor, but use the additional time (almost a full semester) to acknowledge and perhaps modify the potentially limiting goals of the students looking only to pass. Teachers could spend some time and energy on motivating the study of mathematics, by having a strong component of mathematical modeling, and focusing more on numerical solutions to important systems of differential equations. They would have time to include the kinds of activities that engage students and encourage the use of inquiry and other "ambitious teaching methods" [Bressoud, Mesa, and Rasmussen, 2015]. By presenting calculus in a way that illustrates why calculus is important, using examples from a variety of human enterprises and not just in the classical physical and mathematical sciences, the first course in calculus could aggressively invite students into the study of mathematics.

If, in fact, the goals of most high school students taking calculus are focused on college admissions rather than mastery of the content, and if the continuing research supports the premise that learners' goals affect their ability to store and retrieve information, then explicit attention to the students' goals for the course seem necessary. Although the calculus remains unchanged from the course I took nearly 50 years ago, the calculus class, principally the students in that class, are significant different and both sides of the calculus transition must continue moving to address this change.

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Advanced Placement[32] (AP) Calculus:
A Summary of Research Findings Concerning College Outcomes for AP Examinees

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Introduction

Mathematics is a cornerstone of high school education, and it typically forms a part of the general education requirements of college degree programs. Additionally, trends indicate that greater numbers of today's students are pursuing higher level mathematics courses, such as calculus, than in the past. In recent years, calculus courses have become increasingly available in high school, leading students to enter college with advanced mathematics preparation. For many of those students, the preparatory course has been one or both of the Advanced Placement Calculus courses. This paper explores the growth of the AP Calculus course and exam and the associated outcomes of taking these courses. Some of these outcomes include academic performance in college, college retention and completion, and coursework/majors in mathematics, based on research from within the College Board as well as external researchers.

Advanced Placement (AP) is a rigorous academic program built on the commitment, passion, and hard work of students and educators from both secondary schools and higher education institutions. Since 1955, the AP program has enabled millions of students to take college-level courses and exams, and to earn college credit and/or placement while still in high school.

The AP Program has seen tremendous growth over the past decade. A comparison between the public school classes of 2004 and 2014 reveals an increase of more than 500,000 AP examinees. The number of examinees who earned at least one score of 3 or higher rose by over 283,000.[33]

The growth of AP Calculus has mirrored the growth of the AP Program. Among the public school class of 2004, students took 138,124 AP Calculus AB exams, and 39,962 AP Calculus BC Exams, compared to

32 *Advanced Placement* and *AP* are registered trademarks of the College Board. For simplicity, these designations appear in this publication without the registered trademark symbol ®.

33 Source: AP Cohort Data: Graduating Class of 2014. Because these are cohort data (students in the Class of 2014 who took AP Exams at any point in high school), numbers will not align with data for the 2014 AP exam administration (all students who took exams in 2014, regardless of grade level).

234,041 and 84,216, respectively, in the class of 2014.[34] Although the percentage of students scoring 3 or higher has decreased slightly in the past five years, the raw number of students scoring 3 or higher has increased each year. In a study of first-year, first-time students who enrolled in four-year institutions in the fall of 2006, calculus was the most commonly taken mathematics course, more so than algebra, which had been the most commonly taken in Adelman's 2004 study (Shaw and Patterson, 2010). Adelman's study included both two- and four-year institutions, which could have contributed to the larger numbers taking the algebra course.

Studies from the College Board and from external researchers indicate that students who score 3 or higher on AP exams outperform their peers of similar ability and background on various college success outcomes, including academic performance and college retention and completion (Dougherty, Mellor, and Jian, (2006); Geiser and Santelices, 2004; Hargrove, Godin, and Dodd, 2008; Mattern, Shaw, and Xiong, 2009; Patterson, Packman, and Kobrin, 2011).

Academic Performance

Mattern, Shaw, and Xiong (2009) looked at first-year college GPAs across three groups of students: those who took no AP Exams, those who earned a 1 or 2 on the AP Calculus AB Exam, and those who earned a 3 or higher on the AP Calculus AB Exam. The researchers used ANCOVA and logistic regression models, controlling for academic achievement using SAT performance and high school GPA. On average, students who earned a 3 or higher on the AP Calculus Exam had higher first-year college GPAs than students of similar ability who did not take an AP Calculus Exam.

Table 1. Differences in First-Year GPA for AP and No AP Students[35]

Contrast	Point Estimate	Significance	Effect Size
No AP vs. AP Calculus (3, 4, 5)	–0.143	0.000	–0.194

Hargrove, Godin, and Dodd (2008) followed five cohorts of Texas public high school students who attended Texas public colleges and universities. They found that AP Calculus AB Examinees earned first-year GPAs that were significantly higher than non-AP students of similar ability and background.[36] This finding held for fourth-year GPAs as well.

Not only do AP examinees earn higher overall GPAs, on average, they also earn higher subject-area GPAs. According to Patterson, Packman, and Kobrin (2011), whose sample included students at 110 colleges and universities, AP mathematics examinees[37] outperform non-AP students of similar ability and background[38] in their mathematics subject GPA.[39] Expected mathematics subject GPA differences between AP and non-AP students ranged from .196 for exam scores of 3 to .361 for scores of 5.

In addition to GPA, subsequent course performance is another key indicator used to evaluate AP students' level of preparedness for advanced coursework in college. Using propensity score matching, Patterson and Ewing (2013) evaluated whether the AP Exam scores associated with the courses, two of which were AP

34 Source: *AP Report to the Nation:* Class of 2008 Subject-Specific Results for 2004 numbers; AP Cohort Data: Graduating Class of 2014 for 2014 numbers. Because these are cohort data (students in the Classes of 2004 and 2014 who took AP exams at any point in high school), numbers will not align with data for the 2014 AP exam administration (all students who took exams in 2014, regardless of grade level).

35 Source: Mattern, Shaw, and Xiong (2009). Point estimates for retention are measured in odds ratio units. They are the ratios of the odds of lower-ranked groups to those of higher-ranked groups.

36 The study controlled for SAT score and Free and Reduced Price Lunch (FRPL) status. The SAT score was divided into 10 categories, and FRPL into a binary variable, for a 10×2 (20 category) matching design.

37 Includes Calculus AB, Calculus BC, and Statistics

38 Results come from Table 7, Model 3 in the published report. This model controlled for gender, racial or ethnic identity, highest parental education level, high school GPA, SAT Critical Reading score, SAT Mathematics score, and SAT Writing score.

39 Controlling for gender, racial/ethnic identity, socioeconomic status, and prior academic ability.

Calculus AB and AP Calculus BC, were valid for placing students into the subsequent college course related to the exam. The sample included first-time, first-year students entering four-year institutions in fall 2006. For the AP Calculus AB sample, 45 colleges and universities were represented; there were 39 for AP Calculus BC. AP Calculus Examinees who earned credit and thus did not take the introductory college course were matched with non-AP students of similar ability and background who took the introductory college course. The data indicate comparable subsequent course performance among the two groups (standardized difference of 0.173 after matching for AB, and 0.218 for BC). They found that students who earned qualifying AP Calculus AB or BC Exam scores and placed out of colleges' introductory courses performed similarly to their matched[40] non-AP peers in terms of sequent course grade.

Morgan and Klaric (2007) found that AP Calculus students who earned a 3 or higher on the Calculus AB or BC exam had sequent course grades that were estimated to be significantly higher than non-AP students. The logistic regression model controlled for SAT scores.

Table 2. Subsequent Course Grades for AP Calculus AB/BC Examinees and Non-AP Students[41], Before and After Matching[42]

Variable	Before Matching					After Matching				
	AP		Non-AP			AP		Non-AP		
	M	SD	M	SD	d	M	SD	M	SD	d
Course Grade (Calculus AB)	2.974	1.029	2.531	1.183	0.399	2.864	1.085	2.673	1.147	0.173
Course Grade (Calculus BC)	3.130	0.996	2.745	1.108	0.365	2.977	1.048	2.748	1.084	0.218

College Retention and Completion

AP Calculus AB examinees, particularly those who score a 3, 4, or 5 on the exam, are more likely to return for a second year of college compared to students of similar ability who did not take the AP Calculus AB exam. Controlling for SAT performance and high school GPA, Mattern, Shaw, and Xiong (2009) found that the odds of returning for a second year of college were 2.07 times greater for students who earned a 3 or higher on the AP Calculus AB Exam than for students who did not take the AP Calculus AB exam.

In addition to retention findings mentioned above, Hargrove, Godin, and Dodd (2008) found that compared with non-AP students, AP Calculus AB Examinees had significantly higher four-year graduation rates for all four cohorts studied.

Coursework and Majors

Evidence suggests that AP Calculus students, regardless of exam score, take more courses, on average, than non-AP students in disciplines closely related to calculus[43] (Morgan and Klaric, 2007). While non-AP stu-

40 Matching characteristics included gender, racial/ethnic identity, anticipated college major, high school GPA, PSAT/NMSQT® section scores, and mean AP course enrollment at students' high schools.

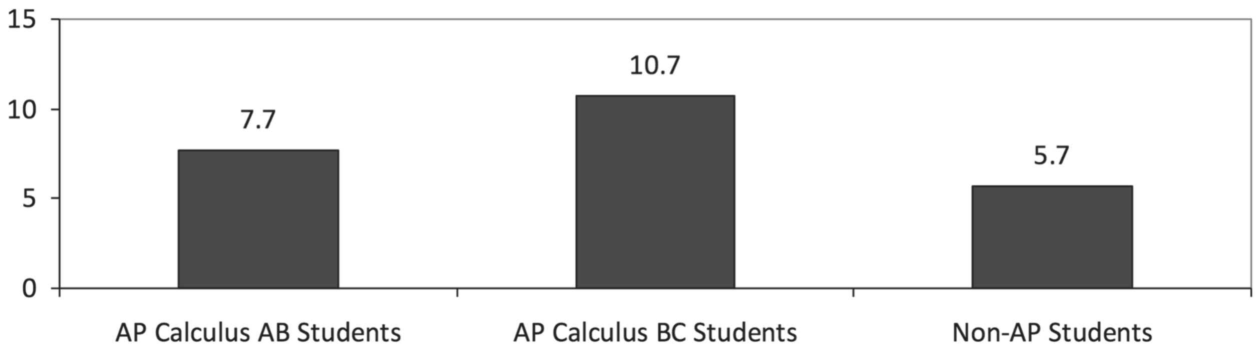
41 AP Examinees included those who took the AP Exam, placed out of the introductory college course, and took a subsequent course in the subject area. Non-AP students took the college's introductory course and a subsequent course in the subject area.

42 Source: Patterson and Ewing (2013). Before matching AP Calculus AB and non-AP Calculus AB examinee group sizes were 3,468 and 6,497, respectively; after matching, both group sizes were reduced to 1,733 across 104 subsequent courses at 45 colleges and universities. For AP Calculus BC, before matching the AP group size was 1,574 and the non-AP group was 6,792; after matching, both group sizes were reduced to 750 across 69 subsequent courses at 39 colleges and universities.

43 Disciplines closely related to AP Calculus included Engineering (Aerospace, Agricultural Ceramic, Chemical, Civil, Computer, Electrical), Applied Mathematics (Mathematics), Civil and Environmental Engineering, Computer Science, Economics and Math,

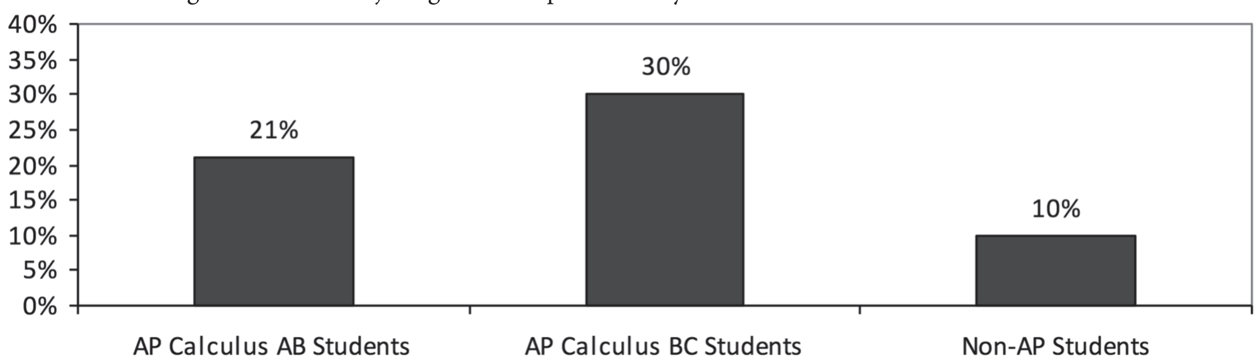
dents take about 5.7 college courses related to calculus, AP Calculus AB students take an average of 7.7 courses and AP Calculus BC students take 10.7 — nearly twice as many courses as non-AP students.[44]

Chart 1. Average Number of College Courses in a Closely Related Discipline[45]



Data also indicate that AP Calculus students are more likely to choose a major in a closely related discipline more often than non-AP students. Morgan and Klaric (2007) highlight the 21% of Calculus AB and 30% of Calculus BC students who major in disciplines related to calculus compared with only 10% of non-AP students. These data are descriptive in nature; no control variables were used.

Chart 2. Percentage of Students Majoring in a Discipline Closely Related to the AP Exam[46]



Tai, Liu, Almarode, and Fan (2010) similarly found that students who took an AP Calculus Exam were more likely to major in a related discipline[47] in college than students who did not take an AP Calculus Exam, controlling for gender, ethnicity, parental education, socioeconomic status, and eighth grade career expectations. In fact, the odds were 4.12 times as high for AP Examinees compared to non-AP students. While this study is quite positive for AP, it does have limitations. The sample is older and cell sizes are quite small at the subject level.

Summary/Conclusions

Multiple studies suggest that there is a positive relationship between AP Calculus Exam participation and college outcomes, including first- and fourth-year GPAs, subject-area GPAs, grades in subsequent courses, time to degree, and majoring in a related discipline. In particular, students who score a 3 or higher on AP Calculus Exams tend to outperform non-AP students of similar ability and background. However, it is

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and Engineering Science and Technology.

44 These data are descriptive in nature; no control variables were used.

45 Source: Morgan and Klaric (2007).

46 Source: Morgan and Klaric (2007).

47 Degree in physical science/engineering concentrations

important to note that these positive associations between AP and college outcomes do not imply causal relationships. There are limitations to the cited research studies, such as not accounting for student motivation, self-selection into AP, or other potential confounding variables. Given the growth in AP Calculus, the AP Calculus AB and BC courses and exams can play a key role in students' transition from high school to college mathematics.

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Factors Influencing Success in Introductory College Calculus

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Introduction

College calculus is the STEM (Science, Technology, Engineering, Mathematics) gatekeeper. For many students, it is the most difficult course they face if they are interested in pursuing a STEM career. While success in college calculus can open the door, failure is a major impediment to continuing to a STEM or pre-med major. The goal of the Factors Influencing College Success in Mathematics (FICSMath) project has been to investigate the factors that contribute to students' success in their first college calculus course, with a special focus on students' mathematics experiences during high school. With this in mind, our team has carried out a study encompassing a large, nationally representative sample of 134 U.S. 2-year colleges, 4-year colleges, and universities. Within these institutions, we collected detailed data from 10,437 introductory calculus students of 336 college calculus instructors. At the end of the semester, instructors reported the final grades of each student. Both a canonical dataset and an extensive codebook have been made available to team members at several institutions, along with their collaborators and graduate students, so that a wide range of research questions could be, and is going to be, investigated. This background chapter presents seven of the most interesting studies generated from the FICSMath project (which relate to students' high school mathematics coursework, including calculus), often using excerpts from papers that we have published.

Methodology

Attempting to collect data that would allow the FICSMath team to investigate as many hypotheses as possible, we scoured the mathematics research literature while also conducting online surveys of high school calculus teachers and college calculus professors. Qualitative analysis helped to group views and explicit hypotheses into themes for which survey items could be constructed. (For details, see Part 1 below.) A focus group with experts in science and mathematics education discussed the items proposed for the survey with an eye to improving clarity and inclusion of relevant options. An initial FICSMath survey was pilot-tested with 47 college calculus students at two local institutions, generating feedback on improving items and adjusting scales. Pilot testing also established that filling out the questionnaire took, on average, 15 to 20 minutes. To ascertain test-retest reliability, we administered the questionnaire twice to 149 students and found high reliability. The final survey containing 61 questions was administered in the fall semester of 2009 during the first few weeks of introductory college calculus classes, with professors reporting grades at the end of the semester.

Sample

For creating our sample, the distinction between 4-year and 2-year institutions served as the first stratification criterion. Each of these two groups thus obtained was further stratified by the size of the institution (small, medium, and large). We used the National Center for Education Statistics (NCES) table of degree-granting postsecondary institutions in the United States, containing fall 2007 enrollment numbers for 2-year institutions and fall 2006 enrollment numbers for 4-year institutions. Size thresholds were set for each of the six stratification cells. Institutions were recruited from the NCES data using a random selection procedure, with bin size first estimated from NCES statistics and refined with calculus enrollment reported by professors teaching calculus from recruited schools. The estimated division is roughly mirrored in our sample, which contains 33.2% students from 2-year schools and 66.8% students from 4-year schools. It also became obvious that many small schools typically do not offer calculus. The bulk of the students taking calculus were enrolled in medium and large institutions. The sample proportions track the proposed bin sizes reasonably well, with the exception that the sample contains a lower percentage of students in medium-size 4-year schools, compared with the percentage of those students in the population (according to our extrapolation): 23.4% vs. 39.9%. Of 276 institutions contacted, 182 (65.9%) initially agreed to participate. In the end, we received usable student questionnaires from 134 (i.e., from 73.6% of those who agreed to participate, or from 48.6% of all contacted institutions). The fact that the questionnaires were administered during class time early in the semester was conducive to a very high student participation rate in each classroom that took part in this study. In addition to this stratified random sample of the national population of institutions of higher education, we over-sampled Hispanic-serving institutions and recruited an additional 55 students from 4 classes/sections in one such institution. Institutions were widely dispersed geographically.

Analysis

The statistical analyses used depended on the research question addressed, but all started with an exploration using descriptive statistics. Quite often, simple counts and proportions revealed interesting patterns. Variables that were highly correlated were often combined into composites using factor analysis, and those composites were standardized for ease of interpretation. While care was taken in the sampling of colleges to produce a nationally representative dataset, there remain differences between the many colleges and courses in terms of grading policies and professors' stringency in the awarding of grades. This situation necessitates an approach, hierarchical linear modeling (HLM), that accounts for the nested nature of the data: students within classrooms (instructors) within institutions. In some cases, propensity methods were used when comparing targeted groups that differed in marked ways. Structural equation modeling provided a way to model the structure of complex or time-dependent relationships, which is beyond the capacity of regression models. Filtering out students with unusual backgrounds (e.g., graduate students taking introductory calculus or students educated outside of the U.S. who had a vastly different mathematics preparation) was often carried out to provide a more homogeneous sample. The large dataset also offered the opportunity to study students that are generally not addressed in smaller studies of high school mathematics (e.g., homeschooled students). In regression models, several demographic controls were employed.

1. Teachers' and Professors' Views of How to Best Prepare Students for College Calculus

Available as: Wade, C. H., Sonnert, G., Sadler, P., Hazari, Z., and Watson, C. (2016). A comparison of secondary mathematics teachers' and mathematics professors' views on secondary preparation for tertiary calculus. *Journal of Mathematics Education at Teachers College, 7*(1), 7–16.

The high school-to-college transition in mathematics is a complex and much-debated issue. Secondary mathematics teachers who teach senior-level students only rarely have the opportunity of examining how well-prepared their students are for subsequent mathematics courses in college, since it is only college professors who can provide such feedback (and the occasional returning student). On the other hand, college

professors are generally unaware of the details of their calculus students' mathematics background. Yet, it has been hypothesized that there are qualitatively contrasting approaches to instruction and divergent views of mathematical thinking between those who teach secondary and college calculus (Hong et al., 2009).
Prior to developing the FICSMath survey questionnaire, we consulted 84 high school mathematics teachers about what they do to prepare their students for college calculus and also asked 185 college mathematics professors about what high school teachers should do to best prepare their students for college calculus. Each was asked to generate up to three statements (some wrote pages). These were categorized into common themes that later informed the construction of items that were included in the FICSMath survey of college calculus students. While teachers and professors showed agreement on many issues, some interesting divergences appeared.

There was agreement among professors and teachers that

- Students need to be supported in the learning of algebra and precalculus, but students must demonstrate their understanding with limited calculator use.
- Pedagogy that incorporates group work can be beneficial, but individual student accountability for content knowledge is also important.
- Placing mathematics in context can support learning.
- Students need to know how to read a mathematics textbook.

The primary areas of discrepancy were

- Professors emphasized that there should be a greater focus on algebra and precalculus in high school and that teaching high school calculus should be deemphasized, while teachers thought that high school calculus provided an opportunity for reviewing and strengthening algebra and precalculus concepts and skills (in addition to learning calculus);.
- Teachers focused heavily upon effective pedagogies for teaching mathematics (e.g., group-work to teach and reinforce mathematics concepts, application to real-world problems), while professors

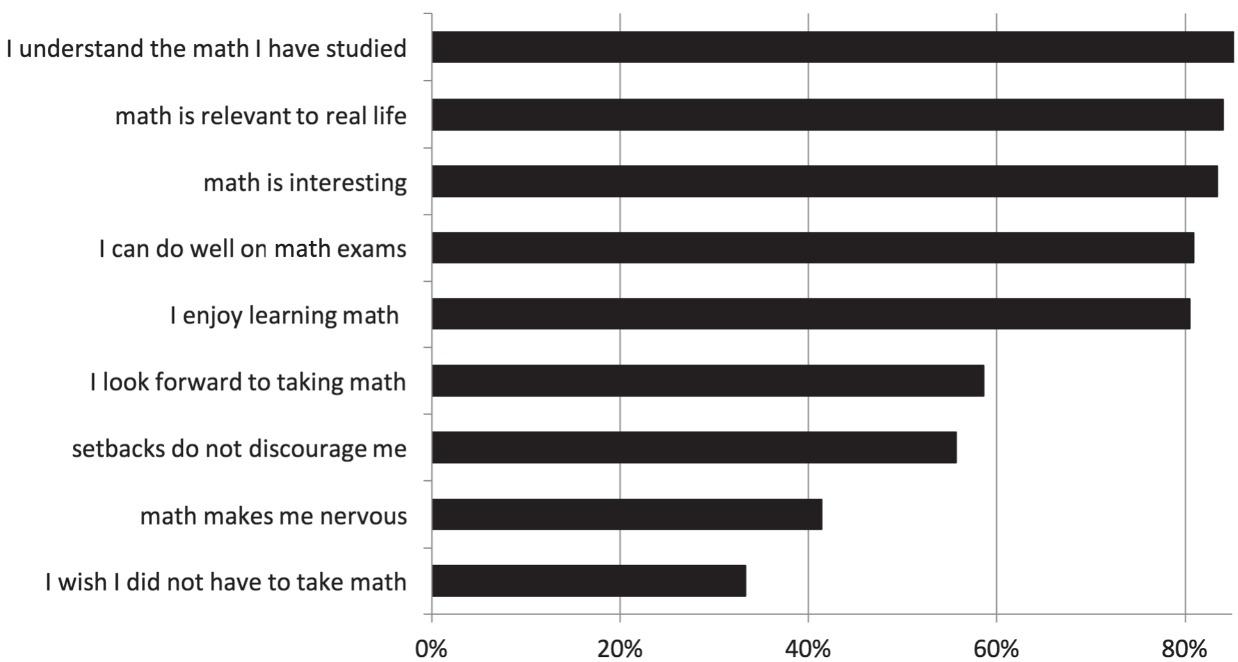

Figure 1. Mathematics Attitudes of College Calculus Students. Percent of students agreeing with the statements listed. Note that the last two state negative attitudes toward mathematics, so that disagreement with these statements indicates a positive attitude. Attitude measures are high, with students feeling that mathematics is interesting, enjoyable, and relevant.

deemed that pedagogy was less important than focusing on the development of deep understanding of mathematics content.

2. Who Takes Introductory College Calculus?

Sadler, P. and Sonnert, G. (2017). The path to college calculus: the impact of high school coursework. *Journal for Research in Mathematics Education.* In press.

As might be expected, students taking college calculus tend to be well prepared in mathematics with a mean SAT quantitative score of 601 (or an ACT score of 26.8). First year students are 48% of the sample. Nine percent self-identify as Hispanic. White students make up 75% of the sample, Black students, 6%, and Asian students, 13%. Males make up 64% of the sample. College calculus students have a variety of career objectives, with 56% aspiring to a STEM career and 20% to a career in health or medicine.

Nearly all students who subsequently enroll in college calculus have had four years of mathematics in grades 9–12. Most (83%) have taken precalculus (with 56% earning an "A" grade) and 52% have taken calculus in high school. By calculating the number of students who move from course to course each year in high school, we produced a diagram denoting the path students follow on their way to college calculus. Clearly evident are two "main sequences" that start with algebra I (Fig. 2):

- One that starts in 8th grade and progresses through geometry, algebra II, and precalculus to calculus or other mathematics courses,
- a second starts with algebra I in 9th grade and progresses through to precalculus in 12th grade.

One in four students reaching high school precalculus appear to skip either a year of geometry or algebra II after algebra I (some may take only a semester of each). In our sample, two-thirds of students who begin algebra I in eighth grade take calculus by the end of high school. Among the one-third of students who start algebra I in ninth grade and then take calculus in their senior year of high school, most take a course that combines algebra II and precalculus or skip a geometry course entirely; i.e., they do not have a full sequence of preparatory math courses prior to high school calculus.

A big surprise was the variety of paths evident, including

- Several years of integrated mathematics in high school.
- Many students with high school calculus retake it in college, even with high AP exam scores.
- Nearly 30% of students take precalculus in college, even though they did well in precalculus or even calculus in high school.

3. Role of Taking Calculus in High School

Sadler, P. and Sonnert, G. (2017). The path to college calculus: the impact of high school coursework. *Journal for Research in Mathematics Education.* In press.

Which is more advantageous: mastery of mathematics preparatory for calculus, or taking calculus in high school? As became evident in our study of teachers' and professors' views about how best to prepare students for college calculus (see above), high school mathematics teachers predictably view all their courses as valuable preparation for further study in college mathematics, yet many college professors have expressed doubts about their worth, and lament students' poor grasp of algebra and precalculus concepts and skills. We modeled the effect of taking high school calculus and of preparation in mathematics considered prerequisite to the study of calculus (i.e., algebra, geometry, precalculus) on later performance in college calculus courses nationwide.

This analysis only examined students who have made it as far a precalculus in high school mathematics in the U.S. (83% of our sample). The taking of high school calculus is a simple dichotomous variable; we did not separate calculus into its many flavors: regular, honors, IB, AP AB, AP BC. The preparation for calculus variable is a normalized composite constructed after a factor analysis that showed a strong relationship between high school grades in non-calculus mathematics (i.e., algebra I, geometry, algebra II, precalculus) and

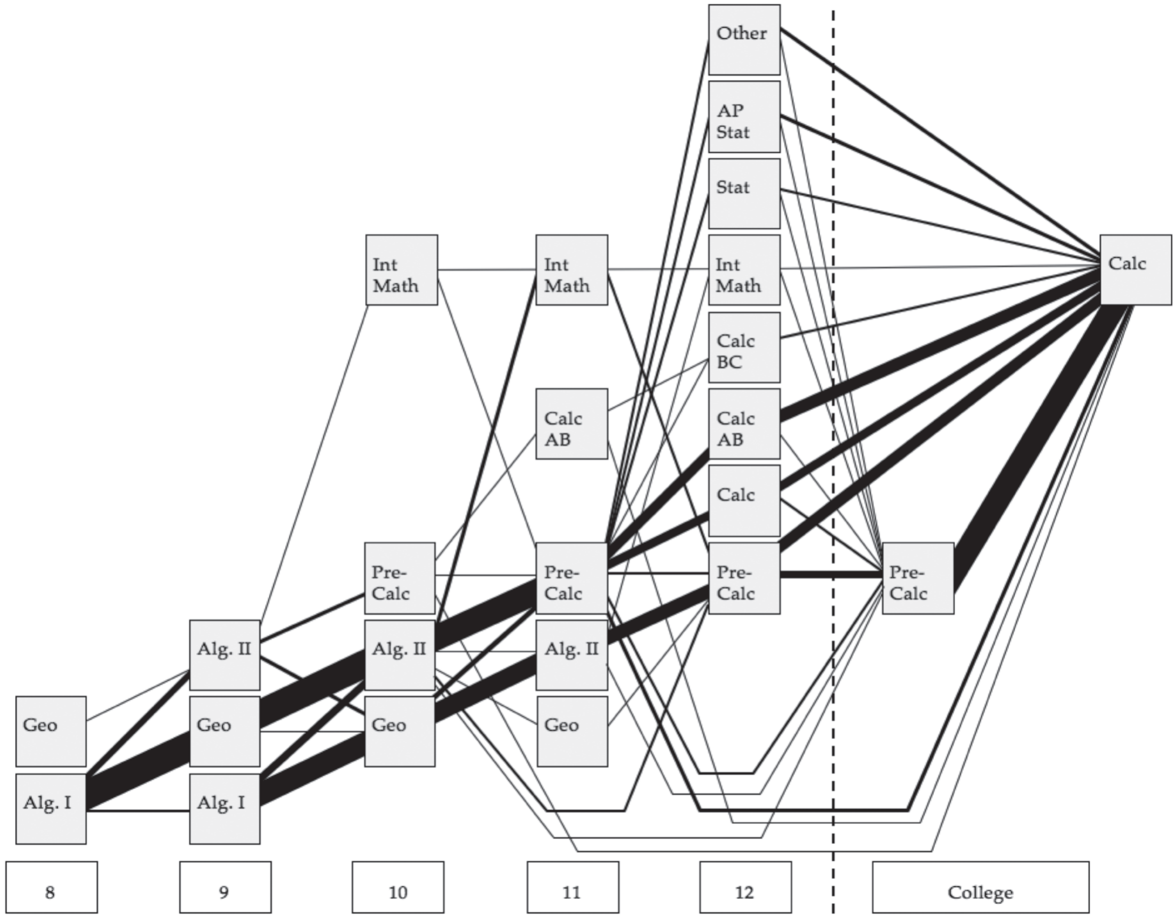

Figure 2. The Path to College Calculus. The yearly sequence, from grade 8 to college, of prior mathematics courses of college calculus students is shown. N = 8,933 students from U.S. high schools. Line thickness is proportional to the number of students transitioning. Note the two major pathways that depend upon when students take algebra I. Note also that 30% of students in introductory college calculus have taken precalculus in college (most of whom who have taken precalculus in high school). Paths of less than 1% of the total are not displayed.

students' SAT or ACT quantitative score.

Based on our regression model, on average, students with a Calculus Preparation Composite greater than –2.0 appeared to benefit from taking a high school calculus course (see Figure 3). Below a HS calculus preparation composite of –2.0, students who took high school calculus did not appear to perform any better in college calculus than those who do not take high school calculus. At the very top of preparation strength (composite ≥ 2.0), the importance of taking high school calculus also faded. Regression models showed that the effect of the preparation for calculus composite was twice as big as the effect of taking a high school calculus course (using the standardized betas of the coefficients), although the impact varied with preparation.

Even students with relatively weak preparation in mathematics appeared to benefit from taking a calculus course in high school. While they may not learn all that much calculus (or earn a high grade), the course can bolster their understanding of concepts and build skills that will be used later in college calculus, supporting the views of high school teachers reported in Wade et. al. (2016; see Section 1 in this chapter), and decrease their chances of needing to take precalculus in college. The result also supports calculus professors' views that a strong background in algebra and precalculus is more important than taking calculus in high school. We take from this that high school students should not be prevented or dissuaded from taking calculus in high school solely based on weak performance in earlier coursework.

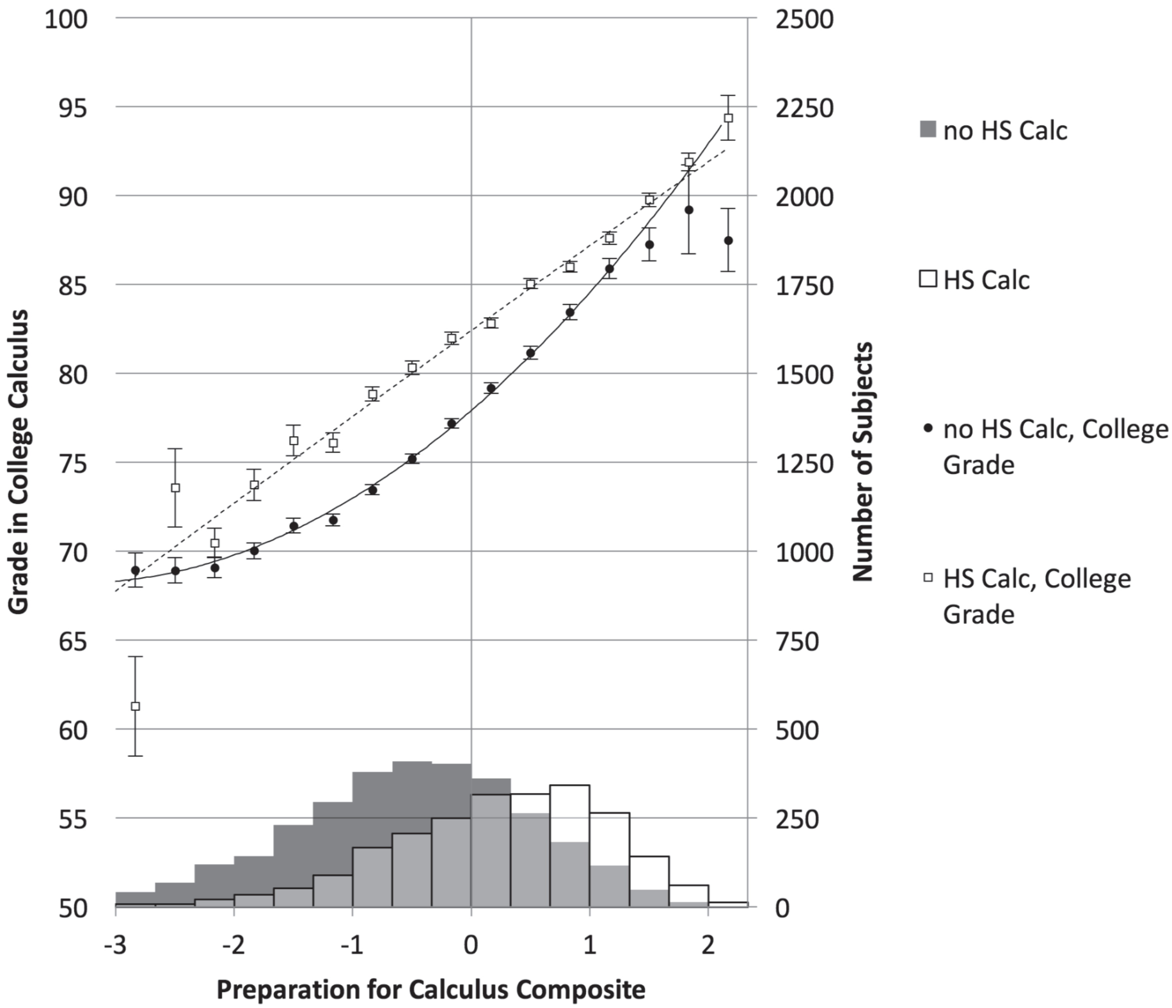

Figure 3. **College Calculus Performance.** Results of a regression model (including college instructor, demographics, and the two variables of interest, along with significant quadratic terms and interactions) are shown, with separate curves for those students taking or not taking high school calculus. Students with a strong preparation for calculus composite were more likely to enroll in high school calculus than those with weaker scores. The mean calculus preparation composite score for those who took high school calculus is 0.23 (SD = 1.03), while who had not taken calculus had a mean of −0.54 (SD = 0.92). Error bars show ±1SE.

4. Productive and Ineffective Efforts

Available as: Barnett, M., Sonnert, G., and Sadler, P. (2014). Productive and ineffective efforts: how student effort in high school mathematics relates to college calculus success. *International Journal of Mathematical Education in Science and Technology, 45*(7), 996–1020.

Deep-seated cultural views proclaim the desirability of effort and project confidence in its efficacy. Hard work and perseverance are firmly rooted in the American psyche as favored ways of overcoming personal limitations (i.e., low initial ability, low social class, etc.). The belief that sheer effort leads to success is enshrined in the proverbial American Dream, which continues to bring hundreds of thousands of immigrants to America, who seek security and prosperity—the hoped-for fruits of their commitment to hard work.

FICSMath data enabled us to test the effects of two types of effort: procedural and intellectual. The following question measured students' procedural effort in their most advanced high school mathematics class:

"How many minutes did you spend reading the textbook both in class and for homework each day on average?" Students reported their reading time in increments of 10 minutes, starting with 0 minutes and capped at "more than 40 minutes." Intellectual effort in the same mathematics class was measured by the following question: "On average, how many minutes did you spend studying or doing work for mathematics outside class each day?" This, too, was reported in minutes. Both variables were normalized for ease of interpretation.

Using a 3-level hierarchical linear model (students within classrooms within institutions) along with demographic variables (gender, race, parental education), HS mathematics performance (SAT/ACT Mathematics score, last HS mathematics grade, took HS calculus), and amount of time spent studying and reading the textbook, we predicted introductory college calculus grade (pseudo $r^2 = 0.290$ in final model). Relevant interactions were investigated as well.

Our findings did not fully support the popular sentiment—among educators, parents, students and society-at-large—about the efficacy of effort. This was especially the case for students' reported time spent reading. We found that, for all students, regardless of their high school mathematics preparation, more time spent reading the mathematics text in high school was associated with worse grades later in college calculus. Our result concerning reading suggests that reading mathematical textbooks is a paradigmatic ineffective effort. Students who spend a large amount of their time reading may do so because they struggle to grasp the course material presented in the classroom and are attempting to gain it by reading. However, spending large amounts of time reading may also be caused by the misfortune of having a weak mathematics instructor in high school. In this case, reading the text is a poor substitute for learning concepts typically covered more skillfully by an experienced and knowledgeable teacher.

The story for study time, by contrast, was not as straightforward as for reading the text: study time was found to be either a productive or an ineffective effort, depending on the students' most advanced high school mathematics course taken and their performance in it. High-performing students experienced productive effort, as did low-performing students enrolled in high school calculus.

In light of our findings, we encourage high school mathematics educators to assign students homework that requires increased study time rather than assignments that revolve around reading the course text. Developing strong study habits prior to attending college was found to reap benefits in college calculus under most circumstances. We suggest instructing all mathematics students in how to effectively study, and beginning with this as early as possible, at least in high school. This is particularly important for students who do not take calculus in high school. Providing instruction on "best study practices" may lead to a stronger mathematical foundation, more positive attitudes towards mathematics, and higher mathematical achievement. An additional approach likely needs to be taken with students who expended considerable study efforts in non-calculus high school courses, but performed poorly in them, especially those who wish to pursue STEM majors in college. These students clearly struggled with the course material in high school, even while dedicating a lot of study time to it in what in the long term turned out to be an ineffective effort. A serious deficit or difficulty in understanding mathematical concepts may have resulted in these students' poor performance in high school despite their extensive efforts, and this problem may have carried over to college and resulted in lower grades in college calculus. Conversely, their peers who received poor mathematics grades in non-calculus classes, while not studying hard, may have possessed better mathematical understanding and skill, but just lacked the commitment to, or interest in, earning higher grades. Simply telling the former group of students to study even more may not foster future success. Rather, these students might need specialized guidance—perhaps extra time going over mathematical concepts and/or effective study methods—to enable them to earn higher mathematics grades in high school and calculus grades in college.

5. Calculator Use in High School

Available as: Mao, Y., White, T., Sadler, P. M., and Sonnert, G. (2016). The association of precollege use of calculators with student performance in college calculus. *Educational Studies in Mathematics.* **dx.doi.org/10.1007/s10649-016-9714-7**

This study investigated how the use of calculators during high school mathematics courses is associated with student performance in introductory college calculus courses. Factor analysis allowed ten items describing high school calculator usage to be reduced to two standardized composites characterizing

1. How extensively calculators were employed.
2. The teachers' restrictions on calculator use.

Hierarchical linear models were used to predict students' college calculus grades, as reported at the end of the term by their professor, while controlling for differences between colleges and in student background. The primary finding was that the more extensively students had used calculators in high school, the lower was their course grade in college calculus. However, some students reported that their high school teachers imposed restrictions on calculator use either by

1. Allowing calculator use only after paper-and-pencil methods were mastered or
2. By limiting the use of calculators to only certain problems on quizzes and exams.

Students whose teachers restricted the use of calculators earned higher college calculus grades. These restrictions were found to offset the negative association of extensive calculator use with college calculus grades. The result resonates with Burrill and Breaux's (2002) suggestion that, while handheld technology should routinely be a part of the learning process of mathematics, the frequency and quality of the use of calculators needs to also be taken into account. Some educators complain that students lack the basic computational and arithmetic skills needed to succeed in higher education, due to calculator dependency (Klein, 2000), and our results are at least partly consistent with this claim, although they do not prove the specific causal mechanism postulated.

6. An Explanatory Model for Mathematics Identity

Available as: Cribbs, J., Hazari, Z., Sonnert, G., and Sadler, P. (2015). Establishing an explanatory framework for mathematics identity, *Child Development*, *86*(4), 1048–1062.

The construct of identity affords researchers the opportunity to explore the association between students' self-perceptions and their persistence in mathematics. Specifically, mathematics identity research can contribute to our understanding of mathematics classroom environments, the broader context of mathematics education, and what it means to be a mathematics learner (Lester, 2007).

A path model was constructed to see how the three hypothesized constituent components of mathematics identity (interest, competence/performance, and recognition) empirically relate to mathematics identity (e.g., degree of agreement with the statement "I am a mathematics person."). Instead of all three components equally predicting mathematics identity, we found that only interest and recognition directly predict mathematics identity, whereas competence/performance exerts an indirect influence on mathematics identity through predicting both interest and recognition.

The effect of competence and performance was strongest on recognition, which indicates that the more strongly students view their ability to understand and do mathematics, the more likely they are to believe that their parents, peers, relatives or teachers see them as a mathematics person, i.e., competence begets recognition. The effect of competence and performance on interest was also significant, indicating that the more strongly students view their ability to understand and do mathematics, the more likely they are to be interested in mathematics. The role of the factor competence and performance is very important, perhaps a prerequisite to interest or recognition, especially considering the positive direct effect that both recognition and interest have on mathematics identity. This suggests that it is not only a high school teacher's role to improve students' competence and performance, but to provide recognition for performance and to stimulate students' interest in mathematics. It is noteworthy that interest and recognition were not that strongly related to each other, and that the path to a mathematics identity may be much more related to the recognition received for competence and performance.

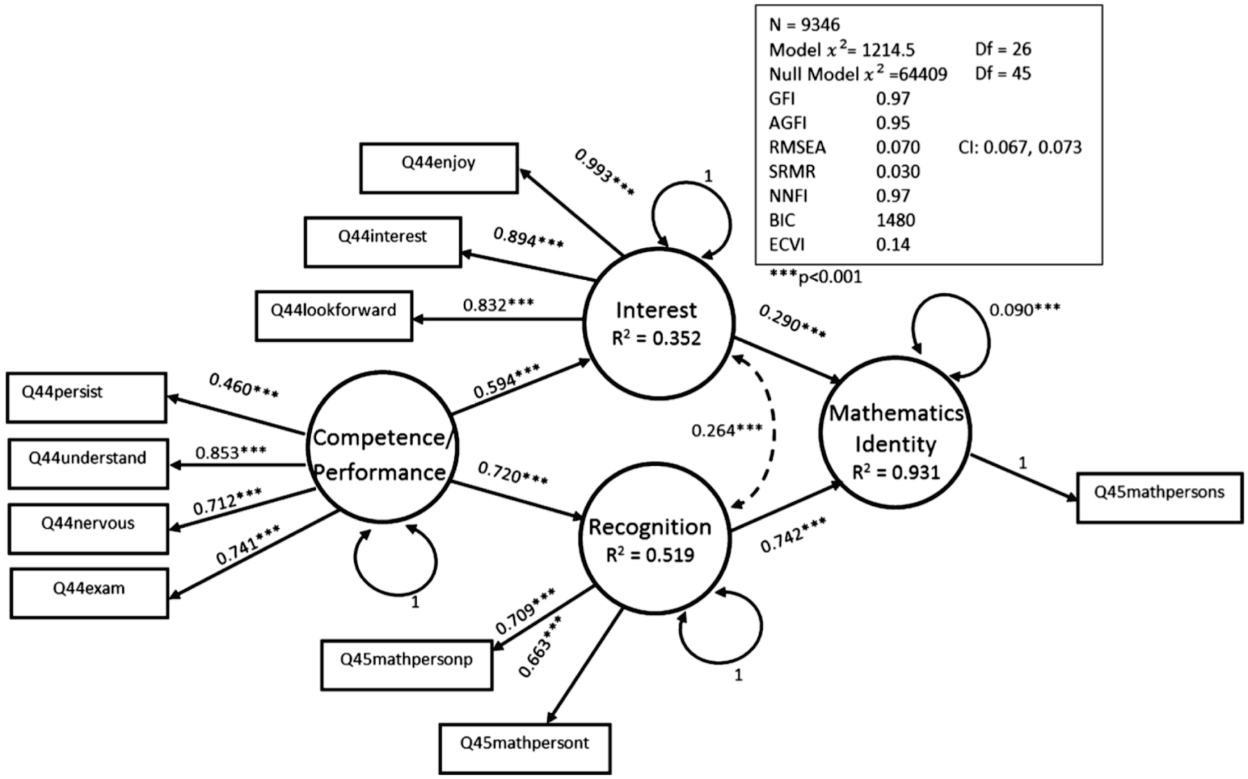

Figure 4. Structural Equation Model for Math Identity of College Calculus Students. GFI = goodness of fit index, AGFI = adjusted goodness of fit index, RMSEA = root mean square error of approximation, SRMR = standardized root mean square residual, NNFI = non-normed fit index, BIC = Bayesian information criterion, ECVI = expected cross-validation index, Df = degrees of freedom, CI = confidence interval.

7. How Effective is College Precalculus in Preparing Students for College Calculus?

Available as: Sonnert, G., and Sadler, P. M. (2014). The impact of taking a college precalculus course on students' college calculus performance. *International Journal of Mathematical Education in Science and Technology*, *45*(8), 1188–1207.

Poor performance on college and university mathematics placement exams keeps many U.S. students who wish to pursue a STEM career from enrolling directly in college calculus. Instead, they must take a college precalculus course that aims to better prepare them for subsequent calculus coursework. In the U.S., enrollment in precalculus courses in 2- and 4-year colleges continues to grow, and these courses are well-populated with students who already took precalculus in high school. This growth in college precalculus is surprising, given the increase nationally in the number of mathematics courses needed to graduate from high school.

We examined student performance in college calculus, using

1. Propensity methods to account for differences in mathematics preparation of those who are allowed to proceed to college calculus and those who are diverted to a semester a precalculus, and
2. Discontinuity regression to estimate the effects of taking college precalculus or not, particularly any differential in calculus performance of those just below or above the "cut off" determining whether they must take precalculus.

A mathematics preparation composite was constructed based on a hierarchical logistic regression predicting the taking of college precalculus. It combined several measures: SAT/ACT mathematics scores, grade in HS precalculus (if taken), type of HS calculus course (if taken) and AP exam grade (if taken). The

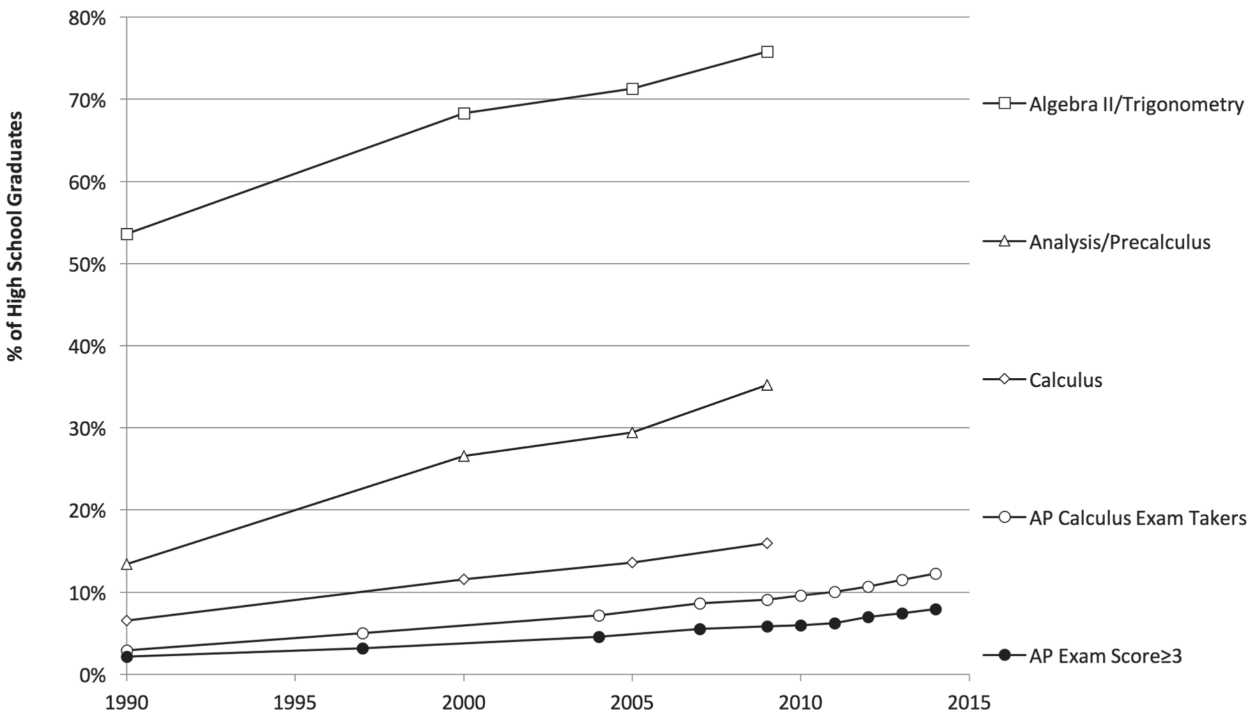

Figure 5. Mathematics Courses Taken by High School Graduates. Note the substantial growth in the percentage of high school graduates having taken advanced mathematics courses. Data sources: high school graduation statistics (Snyder and Dillow, 2013, table 122; Snyder and Dillow, 2015, table 225.40), high school mathematics course enrollments, (National Science Board, 2006, table 1-17, National Science Board, 2014 table 1-8), and reports of AP exam scores.[48]

mean preparation composite for those who went directly to calculus was 0.2, while the mean for those taking college precalculus was –1.1, though the distributions had considerable overlap. We applied propensity methods not to equalize, but to completely separate groups of students, with the goal of using these groups for a discontinuity analysis. This strategy was based on the recognition that the college precalculus course has a specific purpose and a specific target population; its main goal is to help weaker students get ready for calculus. We conducted a sweep of simulated forced assignments of students to either group (because colleges use a variety of tests to measure student readiness for calculus) through the range of their mathematics preparation (modeled by their propensity score) and estimated the calculus grades for students on either side of the cutoff.

Our statistical analysis (by means of a series of discontinuity regressions front-loaded by a propensity measure) yielded no indication that taking a college precalculus course helped the students' subsequent performance in college calculus. Although we simulated a range of scenarios for forced precalculus assignments, none showed a significant benefit of the students' precalculus experience. If anything, there were hints that prior participation in a college precalculus might even be detrimental for the calculus performance of stronger students (i.e., those misplaced in precalculus). There appears to be much room for improvement in college precalculus courses.

Fife (1994) notes that the effectiveness of college precalculus may be compromised by its close similarity to high school precalculus, a course that most students have already taken. Students may not take the college precalculus course seriously or may even be discouraged by having to repeat precalculus. Jarrett (2000) points out that traditional approaches that did not work in high school (i.e., did not prepare students for

48 **research.collegeboard.org/programs/ap/data/archived**, retrieved 7/13/15.

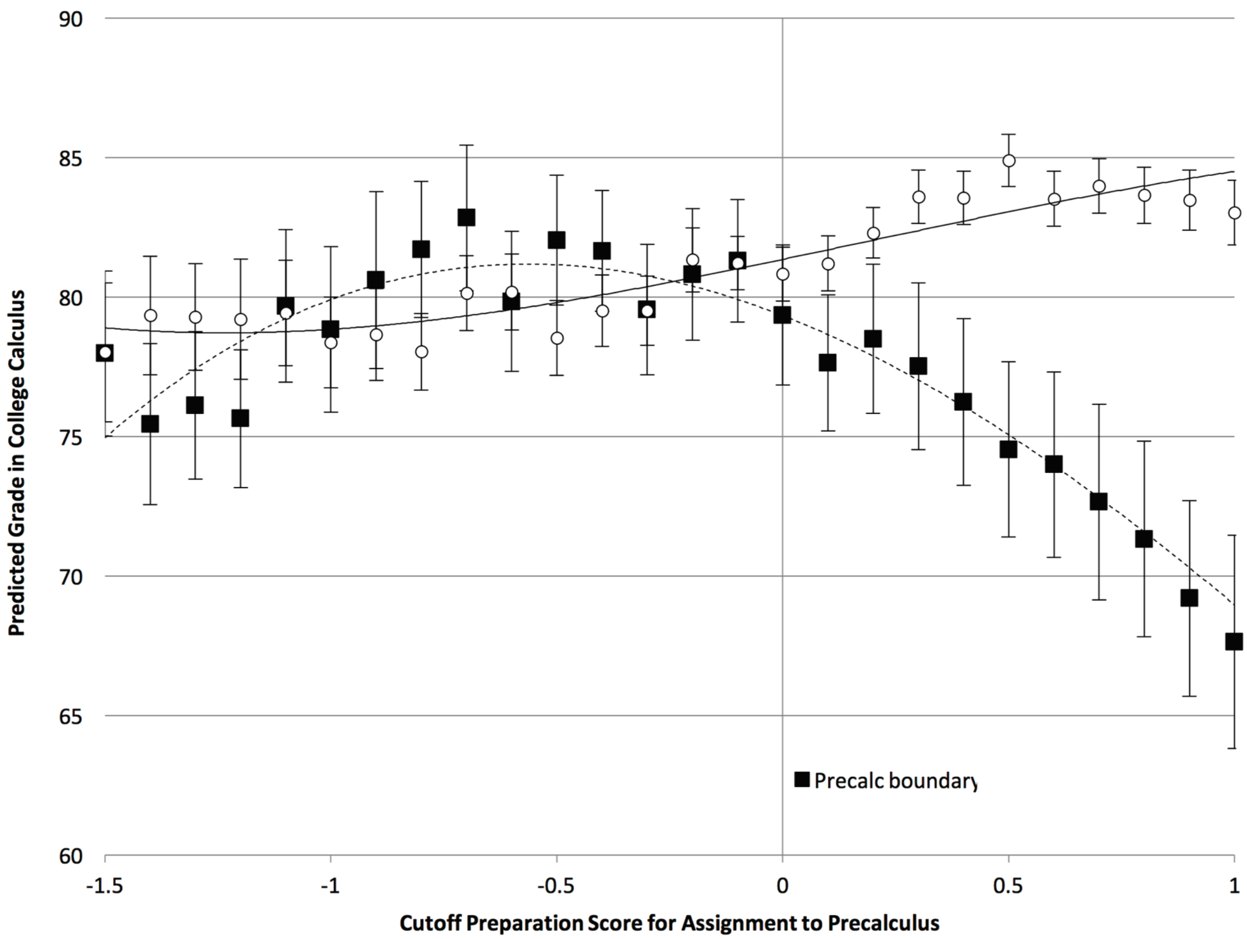

Figure 6. Regression discontinuity gaps in college calculus grade through the range of cut-offs of mathematics preparation for simulated compulsory college precalculus placement. Notes: Error bars indicate ±1 SE. Quadratic regression lines for the cut-off scores of college precalculus takers and non-takers are shown.

doing well on the college mathematics placement exam) are simply repeated in college precalculus (e.g., unrealistic word problems, excessive abstraction)—with no better results. Integrating precalculus with calculus into a single course has long been discussed as an alternative to stand-alone precalculus courses and might be a promising strategy (e.g., Schattschneider, 2006; Tucker, 1996). In any case, preparing students for success in college calculus deserves to become a higher priority for mathematics instructors, education researchers, and policy makers.

Summary

Introductory college calculus has an outsized influence on who can enter the STEM professions, with implications for our nation's economic well-being and security. Those who are ill-prepared for the rigors of calculus often perform poorly in this class.

Mathematics professors regularly bemoan the fact that many students in their introductory calculus classes lack the algebra and precalculus knowledge and skills needed for succeeding and trace this problem back to their precollege mathematics instruction. It bears noting that those students who most excelled in high school mathematics may be missing entirely from introductory college calculus classrooms, having "placed out" of first semester calculus (e.g., with high AP exam scores). Among the students in the introductory college calculus classes, those who have taken high school calculus earn a grade half a letter higher, on average, compared with students with a similar precalculus preparation, but without a high school calculus course. The half of college calculus students who have taken calculus in high school generally appear to have gained a deeper understanding of algebra, geometry, and precalculus from their studies.

High school mathematics teachers are faced with a much wider range of students than college faculty and attempt to both remediate students and advance their knowledge and skills in mathematics. Long gone are the days when all students could be expected to follow a standard course sequence; many more options are now possible (e.g., integrated mathematics, statistics, single semester courses). Yet, even though students are taking more mathematics coursework in high school, the NAEP scores of 17 year-olds have not risen. Use of technology (particularly the ubiquitous graphing calculator) is not a panacea for weak students and may even further handicap them. Even students who are studying hard and reading their mathematics textbook nightly may be using ineffective modes of studying and could benefit from belonging to study groups and adopting more productive ways to learn on their own.

Students who enroll in college calculus are not lacking in a mathematics identity. They like mathematics, find it interesting, and believe it is relevant to their lives. Most feel competent in, and recognized for, their mathematics knowledge, often seeing mathematics as a domain they must fully master if they are to attain their career goals. Many may see being placed in a "developmental" course like precalculus in college as a disappointment, and their later performance rarely benefits from an extra semester of review. Other options for strengthening the mathematics preparation of students must be investigated. Reducing the availability of high school calculus courses or tightening entry requirements for them will probably not be productive. One issue that remains to be examined is the degree to which a year of high school calculus can be thought of as the equivalent of an introductory college calculus course.

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Challenges in the Transition from High School to Post Secondary Mathematics

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> "The transition from school to college mathematics is one of the most troublesome in all of U.S. education. More students report difficulty succeeding in college mathematics than in any other discipline, and more students are dissatisfied with their college mathematics than with any other subject. Why is this?" (Madison, 2011).

Indeed, why? The transition from high school to college mathematics has been a concern for decades. Books such as *Transition to College Mathematics* (Demana and Letizel, 1984) or *Transition to College Mathematics and Statistics* (Hirsch et al., 2008) have been written as fourth year courses for high school students not yet ready for precalculus. Twenty-nine states have some form of transition curricula to bolster students' readiness for post secondary mathematics and English (Barnett et al., 2013) including online resources such as Kentucky's *Transition to College Math* (Newman, et al., 2012). The substance of many of these courses and resources in the past has focused primarily on content, in mathematics from reviewing arithmetic to reexamining algebraic manipulations to applications of the binomial theorem. Research on the impact of these courses is thin; one study in West Virginia indicated that a math transition course had no statistically significant effect on improving college readiness (measured by scores on a placement test) and a negative impact on students' likelihood of passing a college threshold mathematics course (Pheatt et al, 2016). And despite the proliferation of such courses, data suggest that anywhere from 30% to 60% of students continue to leave high school unprepared for college mathematics (Jaschik, 2016).

But perhaps we should be looking at the problem from another perspective, at the very least, paying attention to "lurking" variables that potentially might account for some of the challenges and struggles students have in making the transition from high school to college academics. This paper considers several possible alternate explanations in attempting to answer the question Madison raises: Why is this transition so difficult for so many students? One possible explanation might be the difference in the two learning environments.

I. Learning Cultures

Implicit and explicit expectations exist in high schools and in colleges and universities about the way students are engaged in learning mathematics. The discussion below about these expectations does not characterize every classroom but describes trends and policies that exist in enough classrooms to warrant attention as a part of the K–12 school culture, a culture that often clashes with the cultures of post secondary classrooms.

Current Practices

Standard classroom procedures in high schools include beginning class with "warm ups" or "bell ringers" as a way to bring students to attention and to review an old topic, prepare for a test, or support the development of a new one. They include posting the standards that will be covered in that day's lesson on the board and ensuring that students copy these down in their notebooks and giving students graphic organizers, "a visual and graphic display that depicts the relationships between facts, terms, and/or ideas within a learning task." (Teacher Tap) or having them create "foldables" to organize their learning (Casteel and Narkawicz, 2006). These and many other current classroom practices are intended to support student learning and in some cases give students strategies to become independent learners ready for a college classroom, but even when successfully implemented (another story), the transition to a college classroom where such practices rarely exist and are not called out can add to the disconnect.

In some schools, homework policies are explicit about what kind of homework students can be expected to do, for how long and how this homework is to be counted in the final grading. These can be "one hour for each class per week, while college preparatory students should expect two hours" (Barrientos, 2010) to a maximum of 2.5 hours a day for all subjects, but "schools should note that assigning homework that exceeds the upper limit of these time estimates is not likely to result in additional learning gains—and may even be counter-productive (Cooper, Robinson, and Patall, 2006). In some cases these policies forbid any required homework as an intrusion on students' personal time (Burrill, 2015). In contrast, doing homework outside of class is often the space in which college students actually confront the material they are to learn for the first time.

"Standards based grading", mandated in many districts, presents yet another disconnect. In standards based grading, students are assessed on the learning goals and performance standards for the course with one grade/entry given per learning goal. Typically, retakes and revisions are allowed, and students have multiple opportunities to demonstrate their understanding of classroom standards in various ways (Standards Based Grading, 2013). Students from such environments enter college with expectations of being able to fail, learn from their failures, and retake an exam with no penalty for failure.

Many of these policies are designed to support struggling learners not necessarily those intending to enroll in post high school academic programs—but they are part of the culture and expectations on which teachers are evaluated—even sometimes to the extent that compliance determines the salary they will receive. As a consequence, students are often not taught to be independent learners and are unprepared for the difficulty and amount of work in college classes (Appleby, 2006).

Structural Differences

Well-known differences in time available for teaching and learning contribute to this shift in culture. In high school, courses are often taught across an entire year in three two-hour blocks per week or 50 minutes per day for 180 days (and in some places college preparatory courses are given extended time). Students are expected to do "seatwork" in class, where they often work with classmates on mathematical tasks and may or may not have assigned homework. Contrast this with many typical post secondary introductory courses that meet for three hours a week for 15 weeks or maybe four hours a week including a weekly lab. University students are often expected to take advantage of support options from math learning centers or math labs. Spending time in class doing problems that develop understanding is a luxury; homework is a necessity. Emerging evidence suggests that "flipped" classrooms, where students listen to a videotaped lecture outside of class and do problems during class, can have positive effects on learning (Overmyer, 2014). While this might enable students to have access to an environment closer to that in high school classes, more research is needed, particularly with respect to differences in implementation and the impact on those who most need support to learn the mathematics.

Not only are the high school and undergraduate environments different, research suggests that most high school graduates have many inaccurate perceptions about college that lead to poorly developed notions of what to expect when they enroll in a college program (High School to College Transition Brief, 2010). The differences in expectations for learning between high school courses and post secondary courses has been recognized but typically not well conveyed to high school students. In many post secondary classes, students are expected, on their own incentive, to know about and seek out opportunities to clarify or address a learning gap or to engage in collaborative learning with peers instead of engaging in these experiences as part of the routine of the class. In high school, structured support was available primarily for struggling students; only those with serious learning problems were involved in the support systems. In post high school, the culture shifts, and for students to have some of the same learning experiences that were routine in high school, they have to seek out support on their own, which is not something they had to do in high school.

A second possible alternate explanation in attempting to answer the question Madison raises about the difficulty in transition from high school mathematics to college mathematics could be the lack of academic rigor, with the dominating focus in many high school classes on short-term mathematical learning—the objective of the day. Nearly half of ACT-tested high school graduates who earned a grade of A or B in high school Algebra II were not ready for college math (ACT, 2007). An analysis of test scores in the Baltimore area found that for at least 19 high schools, more than half of the students who earned an A or B in an AP class scored at a low level on the Advanced Placement exam (Bowie, 2013). According to the Advanced Placement Calculus guidelines, to be successful in calculus, students should take courses in algebra, geometry, trigonometry, analytic geometry, and elementary functions, with a strong emphasis on reasoning and structure. But the actual content and academic expectations of the courses matter.

II. Mathematical practices

Advanced Placement courses are designed to reflect what university academics consider important to learn in a particular content area, and the College Board carefully monitors this alignment. The revised calculus framework that will be in place for the 2017 advanced placement calculus AB and BC tests has minimal changes in content; L'Hospital's Rule will be part of the Calculus AB exam, and the limit comparison test, absolute/conditional convergence, and alternating series error bound will be part of the Calculus BC exam. These few changes reflect an overall satisfaction with the content alignment of AP calculus to university courses according to a survey of approximately 90 university calculus instructors as well as the input from representatives of 49 institutions at the 2008 AP Calculus Faculty Colloquium. What did change in the framework was the addition of six Mathematical Practices for Advanced Placement Calculus (MPAC). These practices were included because of concerns raised by post secondary institutions that high school students often seemed to come to calculus having learned rote procedures with little understanding.

The MPAC states that students should engage in

- Reasoning with definitions and theorems
- Connecting concepts
- Implementing algebraic/computational processes
- Connecting multiple representations
- Building notational fluency
- Communicating

These practices are not constrained to calculus but are relevant to statistics and precalculus courses as well. If students are to enter calculus prepared to engage in the practices, that preparation should begin prior to their calculus experience. Unfortunately, many courses in high schools do not give students opportunities to think about mathematics in these ways; most of the textbooks and resources do not routinely incorporate such practices in their approach to learning mathematics. The discussion below suggests examples that could support the implementation of the MPACs in mathematics courses prior to calculus.

In order to be comfortable when asked to reason about the conditions necessary to apply the mean value theorem (MPAC 1) and why such conditions are important or to apply a technique relying on an assumption of normality, students ideally should have prior experience in reasoning with definitions and theorems. Such experiences might include deliberate consideration of the conditions necessary for the graph of a function to have an asymptote not just labeling an asymptote, or being encouraged by their high school instructors to conceptualize a way to reason from some starting point without memorizing a lot of details, (i.e., memorize one or two basic trigonometric formulas and reason from this knowledge base to derive others as needed).

To build a robust understanding of the role of definitions and theorems, precursor work in introductory courses might routinely ask students to produce examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures (College Board, 2016).

Upper level high school mathematics should give students opportunities to "connect concepts to their visual representations" (MPAC 4), for example, to consider the difference between a constant rate of change, one that is not constant and one that is 0 (Figures 1, 2, 3).

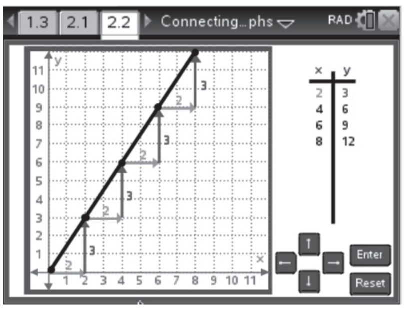


Figure 1. Constant rate

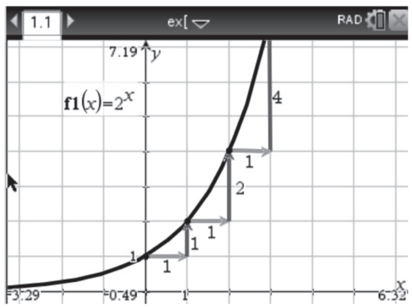


Figure 2. Changing rate

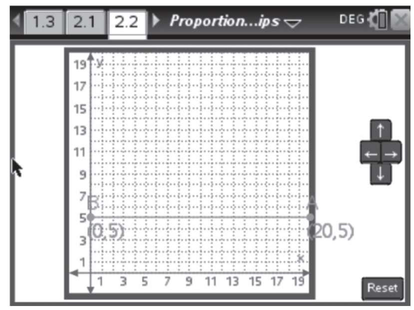


Figure 3. "0" rate of change

Most curricula ask students to graph algebraic relationships and often to look at tables of values but rarely ask students to connect and analyze representations involving numerical and algebraic representations (Example 1).

Example 1

x	$f(x)$	$g(x)$
1	6	2
2	9	3
3	10	4
4	−1	6

For the functions given in the table, if $h(x) = f(g(x)) - 6$,

$h(2) =$

$f^{-1}(6)=$

$g^{-1}(f(1)) =$

The language of calculus is encompassed in a highly refined notational system, which took centuries to develop. Precalculus mathematics should provide students with opportunities to experience tasks that explicitly engage them in thinking about the notation (MPAC 5), for example beginning in early algebra to connect notation to definitions such as representing average rate of change as $\frac{f(b)-f(a)}{b-a}$. They should engage in deliberately connecting notation to graphical, numerical, analytical, and verbal representations (Figures 4 and 5), as well as "assign meaning to notation, accurately interpreting the notation in a given problem and across different contexts" (MPAC 5).

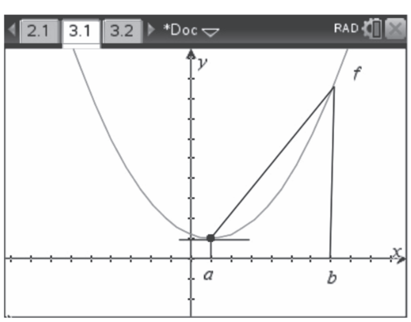


Figure 4. Average rate of change

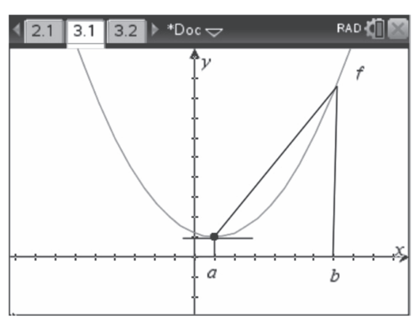
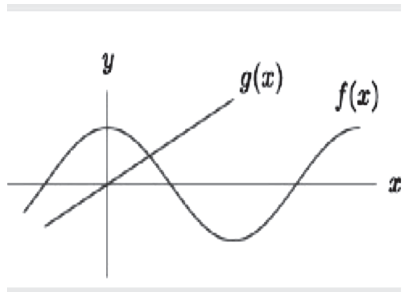


Figure 5. Connecting words to graphs

Experiences should involve students in actively building notational fluency (example 2).

Example 2



Mark specific points to illustrate each statement. If possible label the coordinates of any points you mark.

a) $f(0) = 2$
b) $f(-3) = f(3) = f(9) = 0$
c) $f(2) = g(2)$
d) $g(x) > f(x)$ for $x > 2$

The tasks students do should involve using the connection between concepts (MPAC 2) such as exponents and logarithms or between processes to solve problems (Example 3, figures 6 and 7).

Example 3

$y = \log_2(8x)$ for each positive real number x. Which of the following is true if x doubles:

a) y increases by 3
b) y increases by 2
c) y increases by 1
d) y doubles
e) y triples

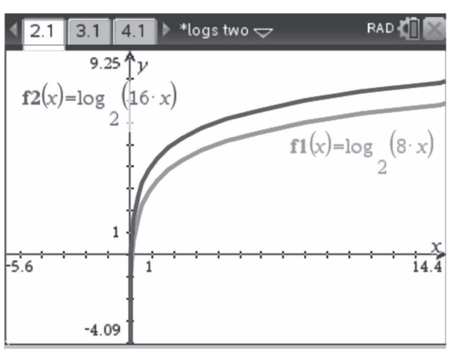


Figure 6. Graphing the problem

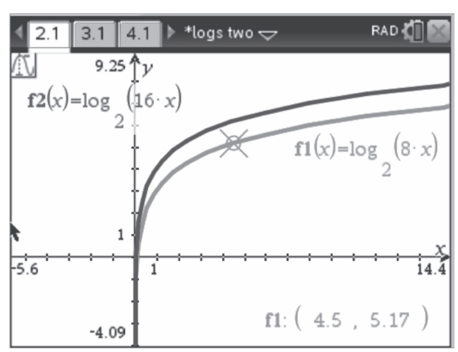


Figure 7. The difference

Resources should help students "identify a common underlying structure in problems involving different contextual situations" (e.g., between "z-scores" in statistics, $\frac{x-\bar{x}}{sd}$ and linear transformations, $y = mx + b$, or between the sum of n whole numbers and the number of handshakes for n people).

Rather than providing dozens of problems where students practice procedures, curricular materials might give students fewer problems but ones that enable them to "establish mathematical lines of reasoning and use them to apply mathematical concepts and tools to solve problems" (College Board, 2016).

Conclusion

The cultures for teaching and learning mathematics in high school and in college have significant differences, and it is not unreasonable that they do. Several steps could be taken to mitigate the transition between the two cultures, however. Dialogue between instructors in the two communities can help high school teachers recognize the constraints and affordances of the university or college system and can help post secondary instructors become aware of and understand their students' expectations for learning in a mathematics class. Learning about and understanding each of the cultures could lead to more informed discussions with students about the transition and help them be ready for the shifts that are likely to happen in the new environment. Perhaps, if high school teachers are aware of these differences and potential harmful consequences for their students in the transition to college mathematics (for example, an overreliance on "retesting"), they might be able to use such information as leverage in negotiating proposed mandates by school administrators more concerned with graduation rates than with learning. University instructors might consider how to incorporate more inquiry and "real time" checking to see whether students are actually following the mathematics.

The transition problem is real. According to a recent survey of calculus students (Sonnert and Sadler, 2014) changes in their mathematics attitudes (i.e., mathematics confidence, enjoyment, and persistence) from the beginning and end of their calculus courses were in the negative direction. Less than 40% of U.S. students who enter post-secondary institutions with an interest in STEM fields finish with a STEM degree. This drops to less than 20% for U.S. minority underrepresented students (Freeman et al, 2014). Perhaps thinking about how to address the shifts in the two cultures can be a step in reversing these findings.

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MAA/NCTM Joint Position on Calculus Adopted March 2012

Question: How should secondary schools and colleges envision calculus as the course that sits astride the transition from secondary to post-secondary mathematics for most students heading into mathematically intensive careers?

MAA/NCTM Position

> While there is an important role for calculus in secondary school, the ultimate goal of the K–12 mathematics curriculum should not be to get into and through a course of calculus by 12th grade, but to have established the mathematical foundation that will enable students to pursue whatever course of study interests them when they get to college. The college curriculum should offer students an experience that is new and engaging, helping to open their understanding of the world of mathematics while strengthening their mastery of tools that they will need if they choose to pursue a mathematically intensive discipline.

In particular, this requires that

1. Students who enroll in a calculus course in secondary school should have demonstrated mastery of algebra, geometry, trigonometry, and coordinate geometry;
2. The calculus course offered in secondary school should have the substance of a mainstream college-level course;
3. The college curriculum should acknowledge the ubiquity of calculus in secondary school, shape the calculus curriculum so that it is appropriate for those who have experienced introductory calculus and offer alternatives to calculus.

Faculty in our colleges and secondary schools should work together to re-envision the role of calculus in secondary and post-secondary mathematics education. From both sides of the transition from secondary to college mathematics, college faculty and secondary teachers should work together to strengthen the mathematical curriculum that is available to students so that those who intend to pursue a mathematically intensive career can acquire the mathematical knowledge and capabilities needed for such a career. They should work to define what it means to be ready for college-level mathematics, assess the effect of college-level mathematics offered in secondary school once students have matriculated at college, and clarify and broaden the understanding of what is meant by college-level mathematics for secondary school. They also need to better understand the mathematical strengths and weaknesses of matriculating students, assess the effectiveness of placement programs for collegiate mathematics, and clarify and broaden the understanding of what the first year of college mathematics can and should entail.

MAA and NCTM are committed to taking appropriate action within the structure of their organizations to assist in guiding the implementation of these recommendations.

Background to the MAA/NCTM Statement on Calculus

David Bressoud[49], Dane Camp[50], Daniel Teague[51]

Evidence of a Problem

In 1984, the United States graduated 113,000 students with a bachelor's degree in engineering, the physical sciences, or the mathematical sciences. By 2010, 112,000 students graduated with a bachelor's degree in one of these disciplines. While there has been some variation over the past quarter century, the number of students earning degrees in these fields has stayed remarkably constant.

Nothing else has remained the same. On the supply side, more students are learning more mathematics in high school than ever before. In 1990, only 14% of high school graduates had completed precalculus or higher and 7% had taken a course in calculus. By 2009, the percentages were 35% and 17%, respectively.[52]

At the other end of the pipe is increased demand for engineers, scientists, mathematicians, and statisticians. In August 2011, President Obama's Council on Jobs and Competitiveness announced a new initiative to increase the number of engineering degrees earned each year by 10,000. As they headlined: "The U.S. Has a Shortage of Engineers, Hindering our Global Competitiveness and Threatening our Ability to Create and Keep High-Tech Jobs." [53] The President's Council of Advisers on Science and Technology (PCAST) published its report in February 2012, calling for one million additional Science, Technology, Engineering, and Mathematics (STEM) degrees over the next decade,[54] including in their evidence a report from the U.S. Department of Commerce that projects a 17% increase in the need for graduates with STEM degrees over this period.[55]

What the members of the mathematical community—especially those in the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM)—have known for a long time is that the pump that is pushing more students into more advanced mathematics ever earlier is not just ineffective: it is counterproductive. Too many students are moving too fast through preliminary courses so that they can get calculus onto their high school transcripts. The result is that even if they are able to pass high school calculus, they have established an inadequate foundation on which to build the mathematical knowledge required for a STEM career. Nothing demonstrates this more eloquently than the fact that from the high school class of 1992, one-third of those who took calculus in high school then enrolled in precalculus when they got to college,[56] and from the high school class of 2004, one in six of those who passed calculus in high school then took remedial mathematics in college.[57]

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52 Table 8 in NCES. America's High School Graduates: Results of the 2009 NAEP High School Transcript Study

53 www.whitehouse.gov/the-press-office/2011/08/31/president-s-council-jobs-and-competitiveness-announces-industry-leaders-

54 PCAST. Report to the President. Engage to Excel.

55 *STEM: Good Jobs Now and for the Future,* July 2011. US Dept. of Commerce. Economics and Statistics Administration.

56 National Education Longitudinal Study of 1988 (NELS:88). nces.ed.gov/surveys/nels88/

57 Education Longitudinal Study of 2002 (ELS:2002). nces.ed.gov/surveys/els2002/

The rush to calculus has had another, more insidious, effect. Today, 61% of the students in college Calculus I have completed a calculus course while in secondary school. Of those, 61% earned an A in their high school class and 30% earned a 3 or higher on an Advanced Placement (AP) Calculus exam. Students who have not had the experience of a high quality calculus course in high school find themselves competing in Calculus I with those who have. At the extreme end of the dysfunctional system that has arisen, there are roughly 5000 students per year who study Calculus AB as juniors, Calculus BC as seniors, and then take Calculus I in their first year of college, essentially taking the same course in three successive years for full credit each time.[58]

Over the past decade a common and often self-fulfilling perception has blossomed among high school and college students that one cannot succeed in college calculus without having first done well in high school calculus. Many otherwise talented students give up on the prospect of a STEM career because they have not had access to a good calculus program in secondary school.

A History of Concern

The Advanced Placement Calculus program began in the 1950s as College Admission with Advanced Standing, an opportunity to provide challenging curricula to talented secondary school students by offering them college-level courses. Calculus made up a very small part of the early exams. The emphasis was on complex and unfamiliar problems that drew on all areas of mathematics. By the late 1950s, AP Calculus offered what was unmistakably a calculus exam. As the program grew, the emphasis shifted from assessing problem-solving ability to testing knowledge of calculus.

Beginning in the 1970s and greatly accelerating in the 1980s, district superintendants and chief state school officers began to see the AP program as a lever by which to raise secondary school quality by challenging the best students to engage in college-level work while still in high school. The hope was that the achievements of the best students would serve to inspire others and so raise the academic level of the entire school.

By 1982, over 200,000 students were studying calculus in high school.[59] Yet most of these classes constituted calculus instruction in name only, a reality reflected in the fact that only 15% of those who enrolled in these classes took the AP Calculus exam.

It was in this atmosphere that MAA directed its Committee on the Undergraduate Program in Mathematics (CUPM) to investigate what was happening in secondary school calculus and to make recommendations. The committee's report, published in 1987,[60] identified five problem areas:

1. Secondary school teacher qualifications and expectations.
2. Student qualifications and expectations.
3. The effect of repeating a course in college after having experienced success in a similar secondary school course.
4. College placement.
5. Lack of communication between secondary schools and colleges.

CUPM found that many students were short-changing their mathematical preparation so that they could enroll in calculus in the 12th grade, and too often this calculus course was nothing like the college course they thought they were experiencing. The committee made seventeen recommendations designed to address these problems. The two most important recommendations were elevated to a joint policy statement of MAA and NCTM, issued in 1986. The two societies recommended that

1. The calculus course offered in the 12th grade should be treated as a college-level course.

58 All data in this paragraph are from the MAA study *Characteristics of Successful Programs in College Calculus.*

59 CUPM Panel, 1987. Report of the CUPM panel on calculus articulation: problems in transition from high school calculus to college Calculus. *The American Mathematical Monthly.* Vol. 94, no. 8, 776–785.

60 Ibid.

2. Students who enroll in a calculus course in secondary school should have demonstrated mastery of algebra, geometry, trigonometry, and coordinate geometry.

In some respects, the situation has improved considerably since this statement was issued. By 2009, over half of the students who studied calculus in secondary school took the AP Calculus exam. An aggressive program of professional development by College Board has reached a large number of teachers. The National Math and Science Initiative is working to introduce quality instruction in calculus to disadvantaged high schools. However, what had started as a means of engaging and challenging our most talented students has turned into the expected curriculum for those students who are heading to college and are able to keep up with an accelerated curriculum in mathematics. By the academic year 2011–12, over 650,000 students were studying calculus in secondary school,[61] close to one-third of the 2,000,000 students who graduate from secondary school and matriculate as full-time college students within a year.

There is now an expectation that every secondary school should offer AP Calculus or its equivalent, with the result that the demand for calculus teachers is outstripping the supply of those who are fully qualified. Within our schools, there is tremendous pressure to fill these classes, accelerating every student who might conceivably be ready for calculus by the senior year regardless of whether such a student might benefit from a slower and more thorough introduction to the traditional topics of high school mathematics. The AP Calculus program continues to grow at around 6% per year.

The problems that were observed in the 1987 report are still with us, now exacerbated by the massive scale of secondary school calculus today. Too many students are being accelerated, short-changing their preparation in and knowledge of algebra, geometry, trigonometry, and other precalculus topics. Too many students experience a secondary school calculus course that drills on the techniques and procedures that will enable them to successfully answer standard problems, but are never challenged to encounter and understand the conceptual foundations of calculus. Too many students arrive at college Calculus I and see a course that looks like a review of what they learned the year before. By the time they realize that the expectations of this course are very different from what they had previously experienced, it is often too late to get up to speed.

The Calculus Bottleneck

There is an even more fundamental problem that was recognized in the 1987 CUPM report. That is the assumption that college-level work in mathematics that is done in secondary school, especially for students heading into science or engineering, must be calculus. The result is that at least one course of calculus before graduation from secondary school has become the norm for the top quartile of college-bound students. It appears that most students who study calculus in secondary school do so not because of a desire to learn calculus, but because their peers are all on this track. The K–12 curriculum is directed toward getting those who are capable of doing so into calculus by grade 12.

This is reinforced by what happens on the other side of the secondary school to college transition.

In the 1960s, CUPM codified what is today the common undergraduate curriculum in mathematics for students heading into science or engineering. It is built upon the assumption that students will study calculus in their first year and that all further mathematics courses may presume that the student has mastered single variable calculus. The unfortunate result is that at most colleges and universities today, a student who intends to major in science or engineering must complete single variable calculus before being allowed into any of the other required mathematics courses, the sole exception being statistics. Calculus has become the great bottleneck of mathematics.

It does not need to be so. The 1987 CUPM report recommended analytic geometry, discrete mathematics, and matrix algebra as alternatives to calculus for those who would study college-level mathematics in 12th grade. We also need to encourage alternatives to calculus for the first year of college. There is no reason for

61 From the NAEP high school transcript study of 2009, 53% of those who studied calculus in high school also took the AP Calculus exam. In 2012, 363,000 took the AP Calculus exam, suggesting that there were 680,000 students who studied calculus in high school.

discrete mathematics or linear algebra or a course in transformational geometry to require calculus as a prerequisite.

More than this, there is real danger in funneling all of our potential science and engineering majors through a double dose of introductory calculus, once on each side of the transition from high school to college. For those students who, for whatever reason, have had a bad experience of calculus in secondary school, the prospect of repeating their experience can dissuade them from continuing their study of mathematics. Even for those who enjoyed their secondary school calculus, beginning college mathematics with a course that looks very much like what they mastered the previous year is at best uninspiring and at worst could lull them into believing that this second iteration of introductory calculus does not require significant effort.

It is no wonder that even well prepared students are put off by their experience of mathematics in college. The college curriculum should offer students an experience that is new and engaging, helping to open their understanding of the world of mathematics while strengthening their mastery of tools that they will need if they choose to pursue a mathematically intensive discipline.

The New Recommendations: A Vision of Secondary and Collegiate Collaboration

On both sides of the transition from secondary to college mathematics, college faculty and secondary teachers must work together to strengthen the mathematics curriculum so that those students who intend to pursue a mathematically intensive career can acquire the mathematical knowledge and capabilities needed for such a career. In March 2012, MAA and NCTM adopted a joint policy statement with the following three recommendations of which the first two reiterate those of 1986.

Recommendation 1: Students who enroll in a calculus course in secondary school should have demonstrated mastery of algebra, geometry, trigonometry, and coordinate geometry.

A common refrain of mathematics faculty in our colleges and universities is that students who fail calculus do so not because they have not been able to learn the calculus but because of a more fundamental lack of the skills and understandings of precalculus. Mastery of algebra, trigonometry, exponentials and logarithms, and an understanding of the role of function in linking covarying quantities are all essential ingredients for the study of calculus. We need tools for assessing student readiness for calculus. We also need strong alternatives to calculus for our secondary schools. The student who is skilled in algebraic and geometric thinking is far better prepared for university-level mathematics than one who has memorized techniques for differentiation and integration.

Recommendation 2: The calculus course offered in secondary school should have the substance of a mainstream college-level course.

We now have evidence that simply having studied calculus in high school confers little or no advantage to students when they enter the first calculus course in college. Benefits do not appear until the student has learned the subject well enough to earn a 3 or higher on an AP Calculus exam.[62] We also know that students who have taken calculus in high school enter college with an inflated sense of their ability to handle university-level mathematics.[63] We need clear guidelines for what is meant by "college calculus in high school" and access to data on which calculus courses are or are not preparing students for university-level mathematics. This includes guidelines for dual enrollment programs, which often have been shown to provide inadequate preparation.[64]

62 Phillip Sadler. 2012. Factors influencing STEM preparedness: from algebra to calculus. *NCTM 2012 Research Presession*.

63 From the MAA study *Characteristics of Successful Programs in College Calculus:* Of the students in mainstream Calculus I who had studied calculus while in high school, over 60% received an A in their high school class and almost two-thirds expected to get an A for their college calculus class. In fact, only a quarter of the students in Calculus I who had taken calculus in high school received an A in the college class.

64 See David Bressoud. *The Dangers of Dual Enrollment.* www.maa.org/external_archive/columns/launchings/launchings_07_07.html

Recommendation 3: The college curriculum should acknowledge the ubiquity of calculus in secondary school, shape the college calculus curriculum so that it is appropriate for those who have experienced introductory calculus in high school, and offer alternatives to calculus.

Colleges and universities can no longer pretend that they can teach calculus the same way that they did in 1990. First of all, the students who twenty years ago made up the top third of the students in Calculus I now skip this course when they enter college. Second, most students enter college calculus already familiar with many of the standard techniques and procedures and a strong preconception of what this course is about and what is required in order to succeed. They are primed to ignore the conceptual development of the subject. Third, the push to accelerate high school students means that many of these students enter college with a weaker mathematical foundation than they would have had a generation ago. And finally, revisiting material they have already studied is not an effective means of engaging students and building a desire to learn mathematics. There are many ways of handling these problems. One is to make it clear that this is not their high school calculus class, either by taking a modeling approach that focuses on differential equations or going to the other extreme and making this an introductory course in analysis. One also can start students with a course that is not calculus, such as discrete mathematics or linear algebra.

Conclusion

The United States has fallen into a seriously dysfunctional system for preparing students for careers in science and engineering, guaranteeing that all but the very best rush through essential parts of the mathematics curriculum and then are forced to sit and spin their wheels while they try to compensate for what was missed. It will take time and work by all involved to repair the transition from high school to college. We cannot afford to wait.

Schedule for the NSF/MAA Workshop on The Role of Calculus in the Transition from High School to College Mathematics

Thursday, March 17

2:00–2:30 Welcome and introductions

2:30–3:30 Opening presentation by Joan Ferrini-Mundy

3:40–4:20 Presentation by Joe Rosenstain (via Skype)

4:30–5:30 Presentation by Ben Hedrick and Sarah Leonard

5:30–7:30 Dinner

7:30–8:45 Guided discussion of issues of concern, David Bressoud

Friday, March 18

8:30–9:00 Breakfast

9:00–9:10 Overview of day's agenda

9:10–9:50 Presentation by Dan Teague

9:55–10:40 Presentation by Gail Burrill

10:40–11:00 break

11:00–11:40 Presentation by Jon Star

11:45–12:25 Presentation by Vilma Mesa and Joe Champion

12:30–1:40 Lunch

1:45–2:25 Presentation by Phil Sadler

2:30–4:30 Work in pre-assigned groups on the questions:

1. What do we know and how do we know it?
2. What do we need to know?

4:40–5:45 Report out, with discussion of which are the most important questions for future research

Saturday, March 19

8:30–9:00 Breakfast

9:00–9:30 Recap of results of Friday with identification of questions on which group would like to work

9:30–11:30 Work in self-selected groups on refining what we need to know and beginning to identify strategies for learning it

11:30–12:30 Report out. Discussion of priorities and next steps

List of Participants

Name	Affiliation
Melissa Barnett	*MIT*
David Bressoud	*Macalester College*
Diane Briars	*NCTM*
Gail Burrill	*Michigan State University*
Joe Champion	*Boise State University*
Ted Coe	*Achieve*
Lloyd Douglas	*MAA*
Peter Ewell	*NCHEMS*
Michael Feder	*STEMx, Batelle*
David Foster	*Silicon Valley Mathematics Initiative*
Joan Ferrini-Mundy	*NSF*
Jessica Fulton	*Chicago Public Schools*
Maya Garcia	*DC Public Schools*
David Goodrich	*DC Public Schools*
Benjamin Hedrick	*College Board*
Brit Kirwan	*University of Maryland*
Erin Krupa	*Montclair State University*
Sarah Leonard	*College Board*
Jim Lewis	*NSF*
Vilma Mesa	*University of Michigan*
Lynn Narasimhan	*DePaul University*
Joe Rosenstein	*Rutgers University*
Dixie Ross	*National Math and Science Initiative*
Phil Sadler	*Harvard Smithsonian Center for Astrophysics*
Heidi Schweingruber	*NRC Board on Science Education*
Gerhard Sonnert	*Harvard Smithsonian Center for Astrophysics*
Jon Star	*Harvard University*
Kathy Stumpf	*Brookhill Institute of Mathematics*
Dan Teague	*North Carolina School of Science and Mathematics*
Debby Ward	*Maryland State Department of Education*

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