George Pólya Awards

Adam Hammett

"Euler's Limit and Stirling's Estimate," *The College Mathematics Journal*, 51:5, 330–336.
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One is always fascinated by the definition of e through Euler's limit, $\lim_{n\to\infty}(1+1/n)^n = e$. It is not easy to prove that this limit exists if one does not use l'Hôpital's rule… And how many of us vaguely know about Stirling's estimate $n! \approx \theta\sqrt{2\pi n}^{1/2}(n/e)^n$? Those of us who do, may avoid mentioning Stirling's estimate in the classroom because of the difficulty of proving it, despite the fact that this estimation is used to calculate $n!$ in software packages for values of n on the order of 100. The paper, "Euler's Limit and Stirling's Estimate," precisely addresses this gap and proposes a natural path to discuss these questions in a calculus course.

The proof of the formulas goes through an analysis of the function $f_c(x) := (1+1/x)^{x+c}$ for $c \in \mathbb{R}$ and positive x. This study highlights the special role played by the value $c = 1/2$, which is needed for Stirling's estimate. To be precise, Hammett proves that $\lim_{n\to\infty} n!/n^{1/2}(n/e)^n$ is a positive constant K. Determining the precise value $K = \sqrt{2\pi}$ is not addressed in the paper, since this cannot be done by elementary means. Hence, a significant merit of the paper is that it separates the elementary part of Stirling's formula from the non-elementary one.

By considering $g_c(x) = \ln f_c(x)$, it is easy to show that f_c is decreasing for $c > 1/2$, and increasing for sufficiently large c when $c < 1/2$. This experimental fact leads to the natural question: *What happens for* $c = 1/2$? Also, it is straightforward that $\lim_{x\to\infty} g_c(x) = 1$ for all c, which yields Euler's limit when taking $c = 0$.

As for the convergence of the sequence $\{\gamma_{n(c)}\} := \{(n!/n^c)(e/n)^n\}$ to a positive real number, one first observes that the sequence is monotonically increasing for $c < 1/2$ and monotonically decreasing for $c > 1/2$. A finer analysis is needed in the particular case $c = 1/2$: the author uses an elegant trick to show that $\gamma_{n(1/2)}$ is both monotonically decreasing and bounded from below by a positive constant through the use of the trapezoid rule for approximation of integrals.

The paper is well-written, clear and entertaining. It presents in an elementary way some deep and important results of analysis that are too often left aside because they are believed to be difficult to present. This paper should be very readable by students as a nice application of calculus and introductory analysis.

Response

This is truly humbling for me. I love *The College Mathematics Journal*, and consider it to be one of the most important publications for college mathematics educators. On numerous occasions, I have been enriched in my own thinking and classroom preparedness because of a featured article. My students and I have mutually benefited in tremendous ways because of this journal, and so to be recognized as having contributed significantly to its content is an honor that I do not take lightly. Thank you.

This particular article's entire content flowed from a seemingly trifling question: *Does the Stirling sequence* $\left\{\frac{n!}{\sqrt{n}}\left(\frac{e}{n}\right)^n\right\}$ *tend to its limit monotonically*? If it were, say, monotonically decreasing (which turns out to be true), then writing down the inequality for consecutive terms of the sequence,

$$\frac{(n+1)!}{\sqrt{n+1}}\left(\frac{e}{n+1}\right)^{n+1} < \frac{n!}{\sqrt{n}}\left(\frac{e}{n}\right)^n,$$

and rearranging and canceling leads us to the equivalent inequality $e < (1+1/n)^{n+1/2}$. This gave me the idea to consider the monotonicity of the function family $(1+1/x)^{x+c}$ for various $c \in \mathbb{R}$, which was just the ticket.

For me this was a lesson, yet again, that we need to keep asking questions and tinkering. You never know what may come of it!

Biographical Sketch

Adam Hammett earned his PhD in combinatorial probability from The Ohio State University in 2007. Ever since he has taught at the college level, and currently serves as professor of mathematics at Cedarville University, where he has been since 2015. He enjoys overseeing undergraduate research projects, reading, and spending time outdoors with his wife Rachael and their four children Isabelle, Madison, Daniel, and Esther.